Joseph Morrison

Reprinted from the Monthly Notices of the Royal Astronomical Society, *Vol. XLIII. No. 7.*

ON THE

COMPUTATION OF THE ECCENTRIC ANOMALY, EQUATION OF THE CENTRE AND RADIUS VECTOR OF A PLANET, IN TERMS OF THE MEAN ANOMALY AND ECCENTRICITY.

BY

J. MORRISON, M.D., M.A.,

Assistant on the American *Ephemeris and Nautical Almanac*, Washington, D.C.

On the Computation of the Eccentric Anomaly, Equation of the Centre and Radius Vector of a Planet, in Terms of the Mean Anomaly and Eccentricity. By J. Morrison, M.D., M.A., Assistant on the American *Ephemeris and Nautical Almanac*, Washington, D.C.

(*Communicated by the Foreign Secretary.*)

Several methods of computing the eccentric anomaly of a planet from the eccentricity and mean anomaly have already been given, in the *Monthly Notices* and elsewhere, by many very able mathematicians. Some of these methods involve considerable labour even after the formulæ have been deduced, while others give only rough approximations. The following method of treating the subject is due, I believe, to the late Professor Hansen, as indicated by him in the *Abhandlungen der Sächsischen Gesellschaft der Wissenschaften*, Band II. It gives the eccentric anomaly with great accuracy and facility by means of rapidly converging series, and in most cases with very little labour, especially after the coefficients of the several terms of the series have been computed for each planet. I purpose in this short paper to develop the subject more fully than is done in the work referred to, and to apply the results to the computation of the eccentric anomaly in the case of each of the primary planets, as well as of the equation of the centre and the radius vector.

Let

$M =$ the Mean Anomaly,

$E =$ the Eccentric Anomaly,

$\nu =$ the True Anomaly,

$e =$ the Eccentricity,

and

$\epsilon =$ the Napierian base.

then we have the following well-known relations:

$$M = E - e \sin E; \tag{1}$$

$$\tan \tfrac{1}{2}\nu = \sqrt{\frac{1+e}{1-e}} \cdot \tan \tfrac{1}{2}E. \tag{2}$$

Hansen assumes

$$y = \epsilon^{E\sqrt{-1}},$$

and

$$z = \epsilon^{M\sqrt{-1}};$$

that is,

$$y = \cos E + \sqrt{-1} \sin E,$$

and

$$z = \cos M + \sqrt{-1} \sin M,$$

and represents the coefficients of the series to be developed according to the sine and cosine of multiples of E by $P^{(i)}_h$ and $Q^{(h)}_i$, thus

$$y^i = \sum_{i=-\infty}^{i=\infty} P^{(i)}_h z^h \tag{3}$$

$$z^h = \sum_{h=-\infty}^{h=\infty} Q^{(h)}_i y^i \tag{4}$$

where h and i are *integers*, and it is evident from the form of the series that we must have

$$P^{(i)}_h = P^{-(i)}_{-h}$$

and

$$Q^{(h)}_i = Q^{-(h)}_{-i},$$

except in the case of $h=0$.

Relation between $P^{(i)}_h$ and $Q^{(h)}_i$.

Put

$$h = i + \mu \text{ in (3)},$$

and we shall have

$$y^i = \sum_{i=-\infty}^{i=\infty} P^{(i)}_{i+\mu} z^{i+\mu}.$$

Multiply by $z^{-\mu}dM$ and integrate between the limits π and $-\pi$, thus

$$P^{(i)}_{i+\mu} = \frac{1}{2\pi}\int_{-\pi}^{\pi} y^i z^{-\mu-i} dM$$

$$= \frac{1}{2\pi\sqrt{-1}}\int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} y^i z^{-\mu-i-1} dz \tag{5}$$

since

$$dM = \frac{dz}{z\sqrt{-1}}.$$

Again, in (4) put $i = h + \nu$, multiply by $y^{-\nu}dE$, and integrate between the limits π and $-\pi$, thus

$$Q^{(h)}_{h+\nu} = \frac{1}{2\pi}\int_{-\pi}^{\pi} y^{-\nu-h}z^h dE$$

$$= \frac{1}{2\pi\sqrt{-1}}\int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} y^{-\nu-h-1}z^h dy \qquad (6)$$

since

$$dE = \frac{dy}{y\sqrt{-1}}.$$

Let

$$i + \mu = h, \text{ and } h + \nu = i;$$

that is

$$\mu + \nu = 0.$$

Then (5) and (6) become

$$P^{(i)}_h = \frac{1}{2\pi\sqrt{-1}}\int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} y^i z^{-h-1} dz, \qquad (7)$$

$$Q^{(h)}_i = \frac{1}{2\pi\sqrt{-1}}\int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} y^{-i-1}z^h dy. \qquad (8)$$

Integrating (7) by parts, we have,

$$P^{(i)}_h = \frac{1}{2\pi\sqrt{-1}}\cdot\frac{i}{h}\int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} y^{i-1}z^{-h} dy, \qquad (9)$$

Comparing (8) and (9), we have

$$P^{(i)}_h = \frac{i}{h}\, Q^{-(h)}_{-i} = \frac{i}{h}\, Q^{(h)}_i,$$

the required relation.

Eliminating E and M from the equations

$$y = \epsilon^{E\sqrt{-1}}, \quad z = \epsilon^{M\sqrt{-1}}, \text{ and } M = E - e \sin E,$$

we get

$$z = y\,\epsilon^{-\frac{1}{2}e\,(y - y^{-1})},$$

which, substituted in (8), gives

$$Q_i^{(h)} = \frac{1}{2\pi\sqrt{-1}} \int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} y^{h-i-1} \epsilon^{-\frac{1}{2}eh(y-y^{-1})} dy \qquad (10)$$

Put

$$\lambda = -\tfrac{1}{2}eh,$$

and develop

$$\epsilon^{\lambda(y-y^{-1})},$$

by the exponential theorem, thus

$$\begin{aligned}
\epsilon^{\lambda(y-y^{-1})} &= 1 + \lambda(y-y^{-1}) + \frac{\lambda^2}{1\,.\,2}(y-y^{-1})^2 + \frac{\lambda^3}{1\,.\,2\,.\,3}(y-y^{-1})^3 + \text{ etc.} \\
&= \left(1 - \frac{\lambda^2}{1^2} + \frac{\lambda^4}{2^2} - \frac{\lambda^6}{6^2} + \frac{\lambda^8}{24^2} - \frac{\lambda^{10}}{120^2} + \ldots\ldots\ldots\right) y^0 \\
&+ \left(\lambda - \frac{\lambda^3}{1\,.\,2} + \frac{\lambda^5}{2\lfloor 3} - \frac{\lambda^7}{6\lfloor 4} + \frac{\lambda^9}{4\lfloor 6} - \ldots\ldots\ldots\right) y \\
&+ \left(-\lambda + \frac{\lambda^3}{1\,.\,2} - \frac{\lambda^5}{2\lfloor 3} + \frac{\lambda^7}{6\lfloor 4} - \frac{\lambda^9}{4\lfloor 6} \ldots\ldots\ldots\right) y^{-1} \\
&+ \left(\frac{\lambda^2}{\lfloor 2} - \frac{\lambda^4}{\lfloor 3} + \frac{\lambda^6}{2\lfloor 4} - \frac{\lambda^8}{\lfloor 6} + \frac{\lambda^{10}}{24\lfloor 6} - \ldots\ldots\ldots\right) y^2 \\
&+ \left(\frac{\lambda^2}{\lfloor 2} - \frac{\lambda^4}{\lfloor 3} + \frac{\lambda^6}{2\lfloor 4} - \frac{\lambda^8}{\lfloor 6} + \frac{\lambda^{10}}{24\lfloor 6} - \ldots\ldots\ldots\right) y^{-2} \\
&+ \left(\frac{\lambda^3}{\lfloor 3} - \frac{\lambda^5}{\lfloor 4} + \frac{\lambda^7}{2\lfloor 5} - \frac{\lambda^9}{6\lfloor 6} + \frac{\lambda^{11}}{24\lfloor 7} - \ldots\ldots\ldots\right) y^3 \\
&+ \left(-\frac{\lambda^3}{\lfloor 3} + \frac{\lambda^5}{\lfloor 4} - \frac{\lambda^7}{2\lfloor 5} + \frac{\lambda^9}{6\lfloor 6} - \frac{\lambda^{11}}{24\lfloor 7} + \ldots\ldots\right) y^{-3} \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ \frac{\lambda^m}{\lfloor m}\left(1 - \frac{\lambda^2}{1\,.\,m+1} + \frac{\lambda^4}{1\,.\,2\,.\,m+1\,.\,m+2} \right. \\
&\qquad \left. - \frac{\lambda^6}{1\,.\,2\,.\,3\,.\,m+1\,.\,m+2\,.\,m+3} + \ldots\right) y^m
\end{aligned}$$

where m must be considered a positive integer.

Let us now represent the coefficient of y^m by $J_\lambda^{(m)}$ (Hansen's notation for the Besselian Function); then the above series may be written thus:

$$\epsilon^{\lambda(y-y^{-1})} = \sum_{m=-\infty}^{m=\infty} J_\lambda^{(m)} y^m, \qquad (11)$$

and from the form of the preceding development we notice that

$$J_\lambda^{(-m)} = (-1)^m J_\lambda^{(m)}, \quad J_{-\lambda}^{(m)} = (-1)^m J_\lambda^{(m)} \quad {}_{-\lambda}^{(-m)} = J_\lambda^{(m)}.$$

From the last two equations we have,

$$Q_i^{(h)} = \frac{1}{2\pi\sqrt{-1}} \sum_{m=-\infty}^{m=\infty} J_\lambda^{(m)} \int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} y^{h-i-m-1} dy. \tag{12}$$

Now, since

$$\int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} y^n dy = 0,$$

when n is an integer and

$$\int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} y^{-1} dy = 2\pi\sqrt{-1},$$

we must have $h-i-m=0$, otherwise (12) would vanish; therefore we have

$$Q_i^{(h)} = \frac{1}{2\pi\sqrt{-1}} \sum_{m=-\infty}^{m=\infty} J_\lambda^{(m)} \int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} y^{-1} dy$$

$$= \sum_{m=-\infty}^{m=\infty} J_\lambda^{(m)};$$

hence

$$P_h^{(i)} = \frac{i}{h} \sum_{m=-\infty}^{m=\infty} J_\lambda^{(m)}.$$

But

$$y^i = \sum_{i=-\infty}^{i=\infty} P_h^{(i)} z^h = \frac{i}{h} \sum_{h=-\infty}^{h=\infty} J_\lambda^{(h-i)} z^h;$$

and also

$$y^i = \cos i\mathrm{E} + \sqrt{-1}\sin i\mathrm{E}$$

$$z^h = \cos h\mathrm{M} + \sqrt{-1}\sin h\mathrm{M};$$

therefore we have

$$\cos i\mathrm{E} + \sqrt{-1}\sin i\mathrm{E} = \frac{i}{h} \sum_{h=-\infty}^{h=\infty} J_\lambda^{(h-i)} (\cos h\mathrm{M} + \sqrt{-1}\sin h\mathrm{M}).$$

Equating the real and imaginary parts of this equation we have

$$\cos i\mathrm{E} = \frac{i}{h} \sum_{h=-\infty}^{h=\infty} J_\lambda^{(h-i)} \cos h\mathrm{M} \tag{13}$$

$$\sin i\mathrm{E} = \frac{i}{h} \sum_{h=-\infty}^{h=\infty} J_\lambda^{(h-i)} \sin h\mathrm{M} \tag{14}$$

When $h=0$, the second members of the last two equations assume the form $\frac{0}{0}$; hence it becomes necessary to evaluate them. Putting $h=0$ in (7) it becomes

$$P_0^{(i)} = \frac{1}{2\pi\sqrt{-1}} \int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} y^i z^{-1}\, dz.$$

Differentiating

$$z = y\epsilon^{-\frac{1}{2}e\,(y-y^{-1})}$$

we get

$$z^{-1}\,dz = y^{-1}\,dy - \frac{e}{2}\,(1 + y^{-2})\,dy,$$

which, substituted in the preceding equation, gives

$$P_0^{(i)} = \frac{1}{2\pi\sqrt{-1}} \int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} (y^{i-1} - \frac{e}{2}\,y^i - \frac{e}{2}\,y^{i-2})\,dy$$

$$= 0,$$

for all values of i greater than 1 or less than -1. Therefore, 0, 1 and -1 are the values of i, which make the second members of (13) and (14) take the form $\frac{0}{0}$.

When $i=0$, we have

$$P_0^{(0)} = \frac{1}{2\pi\sqrt{-1}} \int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} (y^{-1} - \frac{e}{2} - \frac{e}{2}\,y^{-2})\,dy$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} y^{-1}\,dy - \frac{1}{2\pi\sqrt{-1}} \cdot \frac{e}{2} \int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} (1 + y^{-2})\,dy$$

$$= 1 - 0 = 1.$$

When $i=1$, we have

$$P_0^{(1)} = \frac{1}{2\pi\sqrt{-1}} \int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon^{\pi\sqrt{-1}}} (1 - \frac{e}{2}\,y - \frac{e}{2}\,y^{-1})\,dy = \tfrac{1}{2}e.$$

When $i=-1$, we have

$$P_0^{(-1)} = \frac{1}{2\pi\sqrt{-1}} \int_{\epsilon^{-\pi\sqrt{-1}}}^{\epsilon\,\sqrt{-1}} (y^{-2} - \frac{e}{2}\,y^{-1} - \frac{e}{2}\,y^{-3})\,dy = -\tfrac{1}{2}e.$$

For all other values of i we have

$$P_0^{(i)} = 0.$$

From (13) we find, when $i=1$, 2, &c.

$$
\begin{aligned}
\cos E = -\frac{e}{2} &+ \left(1 - \frac{3e^2}{2^3} + \frac{5e^4}{2^6 . 3} - \frac{7e^6}{2^{10} . 3^2} + \frac{e^8}{2^{11} . 5} - \frac{11e^{10}}{2^{17} . 3^3 . 5^2} + \dots\right) \cos M \\
&+ \left(\frac{e}{2} \dots - \frac{e^3}{3} \dots + \frac{e^5}{2^4} - \frac{e^7}{2^2 . 3^2 . 5} + \frac{e^9}{2^7 . 3^3} - \frac{e^{11}}{2^6 . 3^2 . 5^2 . 7} + \dots\right) \cos 2M \\
&+ \left(\frac{3e^2}{2^3} - \frac{3^2 . 5e^4}{2^7} + \frac{7 . 3^4 e^6}{2^{10} . 5} - \frac{3^6 e^8}{2^{13} . 5} + \frac{3^9 . 11e^{10}}{2^{17} . 5 . 7} - \dots\right) \cos 3M \\
&+ \left(\frac{e^3}{3} - \frac{2e^5}{5} + \frac{2^3 e^7}{3^2 . 5} - \frac{2^3 e^9}{3^3 . 7} + \frac{2e^{11}}{3^2 . 5 . 7} - \dots\right) \cos 4M \\
&+ \left(\frac{5^3 e^4}{2^7 . 3} - \frac{5^4 . 7e^6}{2^{10} . 3^2} + \frac{5^6 e^8}{2^{13} . 7} - \frac{5^8 . 11e^{10}}{2^{18} . 3^3 . 7} + \dots\right) \cos 5M \\
&+ \left(\frac{3^3 e^5}{2^4 . 5} - \frac{3^4 e^7}{2^2 . 5 . 7} + \frac{3^6 e^9}{2^8 . 7} - \frac{3^6 e^{11}}{2^7 . 5 . 7} + \dots\right) \cos 6M \\
&+ \left(\frac{7^5 e^6}{2^{10} . 3^2 . 5} - \frac{7^6 e^8}{2^{13} . 5} + \frac{7^9 . 11e^{10}}{2^{18} . 3^4 . 5 . 7} - \dots\right) \cos 7M \\
&+ \left(\frac{2^7 e^7}{3^2 . 5 . 7} - \frac{2^9 e^9}{3^4 . 7} + \frac{2^{12} e^{11}}{3^3 . 5^2 . 7} - \dots\right) \cos 8M \\
&+ \left(\frac{3^{12} e^8}{2^{15} . 5 . 7} - \frac{3^{11} . 11e^{10}}{2^{18} . 5^2 . 7} + \dots\right) \cos 9M \\
&+ \left(\frac{5^7 e^9}{2^8 . 3^4 . 7} - \frac{5^8 e^{11}}{2^7 . 3^3 . 7 . 11} + \dots\right) \cos 10M \\
&+ \left(\frac{11^9 e^{10}}{2^{18} . 3^4 . 5^2 . 7} - \dots\right) \cos 11M \\
&+ \left(\frac{2 . 3^9 e^{11}}{5^2 . 7 . 11} - \dots\right) \cos 12M \\
&+ \dots
\end{aligned}
$$

A check in this and the next series is obtained by putting M=o when the second member reduces to 1 as it should.

$$
\begin{aligned}
\cos 2E = & \left(\quad - e + \frac{e^3}{2^2.3} - \frac{e^5}{2^7.3} + \frac{e^7}{2^9.3^2.5} - \frac{e^9}{2^{14}.3^3.5} + \ldots \right) \cos M \\
& + \left(1 - e^2 + \frac{5e^4}{2^3.3} - \frac{7e^6}{2^3.3^2.5} + \frac{e^8}{2^6.3.5} - \frac{11e^{10}}{2^6.3^3.5^2.7} + \ldots \right) \cos 2M \\
& + \left(e - \frac{3^2e^3}{2^3} + \frac{3^5e^5}{2^7.5} - \frac{3^4e^7}{2^8.5} + \frac{3^6e^9}{2^{14}.7} - \ldots \right) \cos 3M \\
& + \left(e^2 - \frac{2^2e^4}{3} + \frac{2^7.7e^6}{3^2.5} - \frac{2^4e^8}{3.5.7} + \frac{2.11e^{10}}{3^3.5.7} - \ldots \right) \cos 4M \\
& + \left(\frac{5^2e^3}{2^3.3} - \frac{5^4e^5}{2^7.3} + \frac{5^6e^7}{2^8.3^2.7} - \frac{5^9e^9}{2^{15}.3^3.7} + \ldots \right) \cos 5M \\
& + \left(\frac{3^2e^4}{2^3} - \frac{3^4e^6}{2^3.5} + \frac{3^8e^8}{2^7.5.7} - \frac{3^5.11e^{10}}{2^7.5.7} + \ldots \right) \cos 6M \\
& + \left(\frac{7^4e^5}{2^7.3.5} - \frac{7^6e^7}{2^{10}.3^2.5} + \frac{7^8e^9}{2^{15}.3^4} - \ldots \right) \cos 7M \\
& + \left(\frac{2^6e^6}{3^2.5} - \frac{2^{10}e^8}{3^2.5.7} + \frac{2^{12}\,11e^{10}}{3^4.5^2.7} - \ldots \right) \cos 8M \\
& + \left(\frac{3^{10}e^7}{2^{10}.5.7} - \frac{3^{14}e^9}{2^{15}.5.7} + \ldots \right) \cos 9M \\
& + \left(\frac{5^6e^8}{2^7.3^2.7} - \frac{5^8e^{10}}{2^7.3^4.7} + \ldots \right) \cos 10M \\
& + \left(\frac{11^8e^9}{2^{15}.3^4.5.7} - \ldots \right) \cos 11M \\
& + \left(\frac{2.3^5e^{10}}{5^2.7} - \ldots \right) \cos 12M \\
& + \ldots
\end{aligned}
$$

and so on.

Putting $i=1$ in (14), multiplying the resulting series by e, and substituting in (1), we have the following converging series for the direct computation of E.

$$
\begin{aligned}
E = M &+ \left(e - \frac{e^3}{2^3} + \frac{e^5}{2^6 \cdot 3} - \frac{e^7}{2^{10} \cdot 3^2} + \frac{e^9}{2^{18} \cdot 3^2 \cdot 5} - \frac{e^{11}}{2^{17} \cdot 3^3 \cdot 5^2} + \ldots\right) \sin M \\
&+ \left(\frac{e^2}{2} - \frac{e^4}{2 \cdot 3} + \frac{e^6}{2^4 \cdot 3} - \frac{e^8}{2^4 \cdot 3^2 \cdot 5} + \frac{e^{10}}{2^7 \cdot 3^3 \cdot 5} - \frac{e^{12}}{2^7 \cdot 3^3 \cdot 5^2 \cdot 7} + \ldots\right) \sin 2M \\
&+ \left(\frac{3e^3}{2^3} - \frac{3^3 e^5}{2^7} + \frac{3^5 e^7}{2^{10} \cdot 5} - \frac{3^5 e^9}{2^{13} \cdot 5} + \frac{3^7 e^{11}}{2^7 \cdot 5 \cdot 7} - \ldots\ldots\ldots\right) \sin 3M \\
&+ \left(\frac{e^4}{3} - \frac{2^2 e^6}{3 \cdot 5} + \frac{2^2 e^8}{3^2 \cdot 5} - \frac{2^4 e^{10}}{3^3 \cdot 5 \cdot 7} + \frac{2 e^{12}}{3^3 \cdot 5 \cdot 7} - \ldots\ldots\ldots\right) \sin 4M \\
&+ \left(\frac{5^3 e^5}{2^7 \cdot 3} - \frac{5^5 e^7}{2^{10} 3^2} + \frac{5^7 e^9}{2^{12} \cdot 3^2 \cdot 7} - \frac{5^9 e^{11}}{2^{18} \cdot 3^3 \cdot 7} + \ldots\ldots\ldots\right) \sin 5M \\
&+ \left(\frac{3^3 e^6}{2^4 \cdot 5} - \frac{3^5 e^8}{2^4 \cdot 5 \cdot 7} + \frac{3^7 e^{10}}{2^8 \cdot 5 \cdot 7} - \frac{3^8 e^{12}}{2^8 \cdot 5 \cdot 7} + \ldots\ldots\ldots\right) \sin 6M \\
&+ \left(\frac{7^5 e^7}{2^{10} \cdot 3^2 \cdot 5} - \frac{7^7 e^9}{2^{15} \cdot 3^2 \cdot 5} + \frac{7^9 e^{11}}{2^{18} \cdot 3^4 \cdot 5} - \ldots\ldots\ldots\right) \sin 7M \\
&+ \left(\frac{2^7 e^8}{3^2 \cdot 5 \cdot 7} - \frac{2^{11} e^{10}}{3^4 \cdot 5 \cdot 7} + \frac{2^{13} e^{12}}{3^4 \cdot 5^2 \cdot 7} - \ldots\ldots\ldots\right) \sin 8M \\
&+ \left(\frac{3^{12} e^9}{2^{15} \cdot 5 \cdot 7} - \frac{3^{16} e^{11}}{2^{18} \cdot 5^2 \cdot 7} + \ldots\ldots\ldots\right) \sin 9M \\
&+ \left(\frac{5^7 e^{10}}{2^8 \cdot 3^4 \cdot 7} - \frac{5^9 e^{12}}{2^8 \cdot 3^4 \cdot 7 \cdot 11} + \ldots\ldots\ldots\right) \sin 10M \\
&+ \left(\frac{11^9 e^{11}}{2^{18} \cdot 3^4 \cdot 5^2 \cdot 7} - \ldots\ldots\ldots\right) \sin 11M \\
&+ \left(\frac{2 \cdot 3^6 e^{12}}{5^2 \cdot 7 \cdot 11} - \ldots\ldots\ldots\right) \sin 12M \\
&+ \ldots\ldots\ldots
\end{aligned}
$$

Reducing the coefficients of sin M, sin 2M, &c., to logarithms, we have the following series for each of the primary planets.

Mercury.

$$e = \cdot 20560478 \qquad e'' = 42409''\cdot 03$$

$$\begin{aligned} E = M &+ [4\cdot 6251614] \sin M + [3\cdot 6333310] \sin 2\,M \\ &+ [2\cdot 8172458] \sin 3\,M + [2\cdot 0747079] \sin 4\,M \\ &+ [1\cdot 3733623] \sin 5\,M + [0\cdot 6972126] \sin 6\,M \\ &+ [0\cdot 0394355] \sin 7\,M + [9\cdot 3948531] \sin 8\,M \\ &+ [8\cdot 7587952] \sin 9\,M + [8\cdot 1318557] \sin 10\,M \\ &+ [7\cdot 5113252] \sin 11\,M + \end{aligned}$$

For finding the true anomaly we shall have

$$\tan \tfrac{1}{2} \nu = [0\cdot 0905841] \tan \tfrac{1}{2} E.$$

Example. Let $M = 64^\circ\ 10'$.

$$\begin{aligned} E = 64^\circ\ 10' &+ 37969''\cdot 5526 + 3371''\cdot 9202 - 142''\cdot 0962 \\ &- 115''\cdot 5688 - 14''\cdot 9202 + 2''\cdot 1046 + 1''\cdot 0950 \\ &+ 0''\cdot 1114 - 0''\cdot 0349 - 0''\cdot 0132 - 0''\cdot 0008 \\ = 64^\circ\ 10' &+ 11^\circ\ 24'\ 32''\cdot 150 = 75^\circ\ 34'\ 32''\cdot 150 \\ \nu = 87^\circ\ &22'\ 20''\cdot 29. \end{aligned}$$

Venus.

$$e = \cdot 00684331 \qquad e'' = 1411''\cdot 534$$

$$\begin{aligned} E = M &+ [3\cdot 1496888] \sin M + [0\cdot 6839207] \sin 2\,M \\ &+ [8\cdot 3942434] \sin 3\,M + [6\cdot 1783520] \sin 4\,M \end{aligned}$$

$$\tan \tfrac{1}{2} \nu = [0\cdot 0029720] \tan \tfrac{1}{2} E.$$

The Earth.

$$e = \cdot 0167711 \qquad e'' = 3459''\cdot 287$$

$$\begin{aligned} E = M &+ [3\cdot 5389714] \sin M + [1\cdot 4624774] \sin 2\,M \\ &+ [9\cdot 5620725] \sin 3\,M + [7\cdot 7354525] \sin 4\,M \end{aligned}$$

$$\tan \tfrac{1}{2} \nu = [0\cdot 0072842] \tan \tfrac{1}{2} E.$$

Example. Let $M = 71^\circ$.

$$\begin{aligned} E &= 71^\circ + 54'\ 30''\cdot 7060 + 17''\cdot 8574 - 0''\cdot 1987 - 0''\cdot 00527 \\ &= 71^\circ\ 54'\ 48''\cdot 359. \end{aligned}$$

Check.

$$E = M + e'' \sin E$$
$$= 71^\circ + 3288''\cdot359 = 71^\circ\ 54'\ 48''\cdot359.$$

Mars.

$e = \cdot09326113$ $\qquad e'' = 19236''\cdot5$

$$E = M + [4\cdot2836536] \sin M + [2\cdot9515369] \sin 2\,M$$
$$+ [1\cdot7954344] \sin 3\,M + [0\cdot7130829] \sin 4\,M$$
$$+ [9\cdot6715855] \sin 5\,M + [8\cdot6560435] \sin 6\,M$$
$$+ [7\cdot6585193] \sin 7\,M +$$
$$\tan \tfrac{1}{2}\nu = [0\cdot0406208] \tan \tfrac{1}{2} E.$$

Jupiter.

$e = \cdot0482519$ $\qquad e'' = 9952''\cdot67$

$$E = M + [3\cdot9978131] \sin M + [2\cdot3800869] \sin 2\,M$$
$$+ [0\cdot9384309] \sin 3\,M + [9\cdot5706569] \sin 4\,M$$
$$+ [8\cdot2435235] \sin 5\,M + [6\cdot9424833] \sin 6\,M$$
$$\tan \tfrac{1}{2}\nu = [0\cdot0209718] \tan \tfrac{1}{2} E.$$

Saturn.

$e = \cdot0559428$ $\qquad e'' = 11539''\cdot02$

$$E = M + [4\cdot0619992] \sin M + [2\cdot5084302] \sin 2\,M$$
$$+ [1\cdot1309242] \sin 3\,M + [9\cdot8271927] \sin 4\,M$$
$$+ [8\cdot5643061] \sin 5\,M + [7\cdot3274187] \sin 6\,M$$
$$\tan \tfrac{1}{2}\nu = [0\cdot0243210] \tan \tfrac{1}{2} E.$$

Uranus.

$e = \cdot0463592$ $\qquad e'' = 9562''\cdot29$

$$E = M + [3\cdot9804453] \sin M + [2\cdot3453568] \sin 2\,M$$
$$+ [0\cdot8917040] \sin 3\,M + [9\cdot5011044] \sin 4\,M$$
$$+ [8\cdot1567170] \sin 5\,M + [6\cdot8383189] \sin 6\,M$$
$$\tan \tfrac{1}{2}\nu = [0\cdot0201480] \tan \tfrac{1}{2} E.$$

Neptune.

$e = \cdot0089903$ $\qquad e'' = 1854''\cdot38$

$$E = M + [3\cdot2681948] \sin M + [0\cdot9209318] \sin 2\,M$$
$$+ [8\cdot7497592] \sin 3\,M + [6\cdot6523734] \sin 4\,M$$
$$\tan \tfrac{1}{2}\nu = [0\cdot0039045] \tan \tfrac{1}{2} E.$$

Equation of the Centre.

From the theory of elliptic motion we have

$$\frac{d\nu}{dt} = \frac{h}{r^2},$$

where $h=$ twice ~~half~~ the area described in a unit of time; therefore

$$\frac{\mathrm{T}}{\pi ab} = \frac{2}{h};$$

whence

$$h = \frac{2\pi ab}{\mathrm{T}};$$

therefore

$$\frac{d\nu}{dt} = \frac{2\pi ab}{\mathrm{T}r^2},$$

and

$$\frac{d\mathrm{M}}{dt} = \frac{2\pi}{\mathrm{T}}.$$

Now,

$$\frac{d\nu}{d\mathrm{M}} = \frac{d\nu}{dt} \cdot \frac{dt}{d\mathrm{M}} = \frac{ab}{r^2};$$

but

$$b = a\sqrt{1-e^2},$$

and

$$r = a(1 - e\cos\mathrm{E});$$

therefore

$$\frac{d\nu}{d\mathrm{M}} = \frac{\sqrt{1-e^2}}{(1-e\cos\mathrm{E})^2} \qquad (15)$$

Developing the second member by the binomial theorem, and changing the powers of cos E to the cosines of multiples of E, we have

$$
\begin{aligned}
\frac{dv}{dM} &= (1-e^2)^{\frac{3}{2}}\,(1-e\cos E)^{-2} \\
&= \left(1 + e^2 + e^4 + e^6 + e^8 + e^{10} + e^{12} + \ldots\right) \\
&\quad + 2\left(e + e^3 + e^5 + e^7 + e^9 + e^{11} + \ldots\right)\cos E \\
&\quad + \left(\frac{3}{2}e^2 + \frac{7}{2^2}e^4 + \frac{59}{2^5}e^6 + \frac{121}{2^6}e^8 + \frac{491}{2^8}e^{10} + \frac{991}{2^9}e^{12} + \ldots\right)\cos 2E \\
&\quad + \left(e^3 + \frac{11}{2^3}e^5 + \frac{25}{2^4}e^7 + \frac{107}{2^6}e^9 + \frac{223}{2^7}e^{11} + \ldots\right)\cos 3E \\
&\quad + \left(\frac{5}{2^3}e^4 + e^6 + \frac{79}{2^6}e^8 + \frac{89}{2^6}e^{10} + \frac{3073}{2^{11}}e^{12} + \ldots\right)\cos 4E \\
&\quad + \left(\frac{3}{2^3}e^5 + \frac{11}{2^4}e^7 + \frac{59}{2^6}e^9 + \frac{281}{2^8}e^{11} + \ldots\right)\cos 5E \\
&\quad + \left(\frac{7}{2^5}e^6 + \frac{29}{2^6}e^8 + \frac{337}{2^9}e^{10} + \frac{849}{2^{10}}e^{12} + \ldots\right)\cos 6E \\
&\quad + \left(\frac{1}{2^3}e^7 + \frac{37}{2^7}e^9 + \frac{29}{2^6}e^{11} + \ldots\right)\cos 7E \\
&\quad + \left(\frac{9}{2^7}e^8 + \frac{23}{2^7}e^{10} + \frac{155}{2^9}e^{12} + \ldots\right)\cos 8E \\
&\quad + \left(\frac{5}{2^7}e^9 + \frac{7}{2^6}e^{11} + \ldots\right)\cos 9E \\
&\quad + \left(\frac{11}{2^9}e^{10} + \frac{121}{2^{11}}e^{12} + \ldots\right)\cos 10E \\
&\quad + \left(\frac{3}{2^8}e^{11} + \ldots\right)\cos 11E \\
&\quad + \left(\frac{13}{2^{11}}e^{12} + \ldots\right)\cos 12E
\end{aligned}
$$

Substituting the values of cos E, cos 2E, &c., obtained from (13), multiplying by dM and integrating, we have, after putting C, the equation of the centre, $=\nu - M$

$$
\begin{aligned}
C = &+\left(2e - \frac{e^3}{2^2} + \frac{5e^5}{2^5.3} + \frac{107e^7}{2^9.3^2} + \frac{6217e^9}{2^{13}.3^2.5} + \frac{565879e^{11}}{2^{16}.3^3.5^2} + \ldots\right)\sin M \\
&+\left(\frac{5e^2}{2^2} - \frac{11e^4}{2^3.3} + \frac{17e^6}{2^6.3} + \frac{43e^8}{2^7.3^2.5} + \frac{677e^{10}}{2^9.3^3.5} + \frac{7237e^{12}}{2^{10}.3^3.5.7} + \ldots\right)\sin 2M \\
&+\left(\frac{13e^3}{2^2.3} - \frac{43e^5}{2^6} + \frac{95e^7}{2^9} - \frac{973e^9}{2^{12}.3.5} + \frac{19503e^{11}}{2^{16}.5.7} + \ldots\right)\sin 3M \\
&+\left(\frac{103e^4}{2^5.3} - \frac{451e^6}{2^5.3.5} + \frac{4123e^8}{2^8.3^2.5} - \frac{1619e^{10}}{2^7.3^3.7} + \frac{111929e^{12}}{2^{13}.3^3.5.7} + \ldots\right)\sin 4M \\
&+\left(\frac{1097e^5}{2^6.3.5} - \frac{5957e^7}{2^9.3^2} + \frac{164921e^9}{2^{12}.3^2.7} - \frac{4305913e^{11}}{2^{17}.3^3.7} + \ldots\right)\sin 5M \\
&+\left(\frac{1223e^6}{2^6.3.5} - \frac{7913e^8}{2^7.5.7} + \frac{7751e^{10}}{2^{10}.7} - \frac{82021e^{12}}{2^{11}.3.5.7} + \ldots\right)\sin 6M \\
&+\left(\frac{47273e^7}{2^9.3^2.7} - \frac{1773271e^9}{2^{14}.3^2.5} + \frac{93521303e^{11}}{2^{17}.3^4.5} - \ldots\right)\sin 7M \\
&+\left(\frac{556403e^8}{2^{10}.3^2.5.7} - \frac{1182827e^{10}}{2^7.3^4.5.7} + \frac{324319 49e^{12}}{2^{12}.3^4.5.7} - \ldots\right)\sin 8M \\
&+\left(\frac{10661993e^9}{2^{14}.3^2.5.7} - \frac{101836961e^{11}}{2^{17}.5^2.7} + \ldots\right)\sin 9M \\
&+\left(\frac{7281587e^{10}}{2^{10}.3^4.5.7} - \frac{769805651e^{12}}{2^{12}.3^4.5.7.11} + \ldots\right)\sin 10M \\
&+\left(\frac{62929017101e^{11}}{2^{17}.3^4.5^2.7.11} - \ldots\right)\sin 11M \\
&+\left(\frac{7218065e^{12}}{2^{13}.3.7.11} - \ldots\right)\sin 12M
\end{aligned}
$$

Greatest Equation of the Centre.

When the equation of the centre has its maximum value, $d(v-M)=0$ or $dv=dM$; therefore, from (15), we have

$$\cos E = \frac{1-(1-e^2)^{\frac{1}{2}}}{e}$$

$$= \frac{e}{2^2} + \frac{3e^3}{2^5} + \frac{7e^5}{2^7} + \frac{77e^7}{2^{11}} + \dots$$

and

$$\sin E = 1 - \frac{e^2}{2^5} - \frac{49e^4}{2^{11}} - \frac{1233e^6}{2^{16}} - \dots \qquad (16)$$

whence

$$\sin 2E = \frac{e}{2} + \frac{11e^3}{2^6} + \frac{375e^5}{2^{12}} + \frac{7587e^7}{2^{17}} + \dots$$

$$\sin 3E = -1 + \frac{9e^2}{2^5} + \frac{417e^4}{2^{11}} + \frac{1087e^6}{2^{13}} + \dots$$

&c., &c.

Any one of these will give the eccentric anomaly when the equation of the centre is a maximum. Developing

$$\tan \tfrac{1}{2} v = \sqrt{\frac{1+e}{1-e}} . \tan \tfrac{1}{2} E$$

into a series, we have

$$v = E + 2 \tan \frac{\phi}{2} \sin E + \tan^2 \frac{\phi}{2} \sin 2E + \frac{2}{3} \tan^3 \frac{\phi}{2} \sin 3E + \text{ \&c.},$$

where

$$\tan \frac{\phi}{2} = \frac{1-(1-e^2)^{\frac{1}{2}}}{e}$$

$$= \frac{e}{2} + \frac{e^3}{2^3} + \frac{e^5}{2^4} + \frac{5e^7}{2^7} + \text{ \&c.}$$

Substituting the values of sin E, sin 2E, &c., $\tan\frac{\phi}{2}$, $\tan^2\frac{\phi}{2}$ &c., we find

$$v = E + e + \frac{25}{2^5 . 3} e^3 + \frac{1443}{2^{11} . 5} e^5 + \frac{43069}{2^{16} . 7} e^7 + \dots \qquad (17)$$

Multiplying the second series of (16) by e, and substituting in (1), we get

$$M = E - e + \frac{e^3}{2^5} + \frac{49}{2^{11}}e^5 + \frac{1233}{2^{16}}e^7 + \dots \qquad (18)$$

Subtracting (18) from (17), we have

$$C = \nu - M$$

$$= 2e + \frac{11}{2^4 . 3}e^3 + \frac{599}{2^{10} . 5}e^5 + \frac{17219}{2^{15} . 7}e^7 + \dots \qquad (19)^*$$

Reverting this series we have

$$e = \frac{C}{2} - \frac{11}{2^8 . 3}C^3 - \frac{587}{2^{16} . 3 . 5}C^5 - \frac{40583}{2^{23} . 3^2 . 5 . 7}C^7 - \dots \qquad (20)$$

The Radius Vector.

By the properties of the ellipse we have $r = a\,(1 - e \cos E)$, in which substitute the values of $\cos E$; we thus deduce the following series, a check upon which is obtained by putting $M = 0$, when it reduces to $\frac{r}{a} = 1 - e$.

---

* Delambre, in his *Astronomy*, vol. ii. p. 55, has given this series correctly, but farther down on the same page he committed an error in reducing the term in e^7, which he gave as $\frac{17219}{2^{16} . 7}e^7$. He also gave the second term of series (20) incorrectly as $-\frac{11}{2^4 . 3}C^3$. Both of these incorrect values were copied by Baily in his "Astronomical Tables and Formulæ," and from him by several other writers on Astronomy. Mr. Farrel in the *Monthly Notices* of the Royal Astronomical Society, June 1860, corrects Baily as to series (20), but at the same time adopts Delambre's incorrect value of the last term of (19), and computes accordingly.

$$
\begin{aligned}
\frac{r}{a} = & +\left(1 + \frac{e^2}{2}\right) \\
& +\left(-e + \frac{3}{2^3}e^3 - \frac{5}{2^6 \cdot 3}e^5 + \frac{7}{2^{10} \cdot 3^2}e^7 - \frac{1}{2^{11} \cdot 5}e^9 + \frac{11}{2^{17} \cdot 3^3 \cdot 5^2}e^{11} - \dots\right)\cos M \\
& +\left(-\frac{e^2}{2} + \frac{1}{3}e^4 - \frac{1}{2^4}e^6 + \frac{1}{2^2 \cdot 3^2 \cdot 5}e^8 - \frac{1}{2^7 \cdot 3^3}e^{10} + \frac{1}{2^6 \cdot 3^2 \cdot 5^2 \cdot 7}e^{12} - \dots\right)\cos 2M \\
& +\left(-\frac{3}{2^3}e^3 + \frac{3^2 \cdot 5}{2^7}e^5 - \frac{7 \cdot 3^4}{2^{10} \cdot 5}e^7 + \frac{3^6}{2^{13} \cdot 5}e^9 - \frac{3^6 \cdot 11}{2^{17} \cdot 5 \cdot 7}e^{11} + \dots\right)\cos 3M \\
& +\left(-\frac{1}{3}e^4 + \frac{2}{5}e^6 - \frac{2^3}{3^2 \cdot 5}e^8 + \frac{2^3}{3^3 \cdot 7}e^{10} - \frac{2}{3^2 \cdot 5 \cdot 7}e^{12} + \dots\right)\cos 4M \\
& +\left(-\frac{5^3}{2^7 \cdot 3}e^5 + \frac{5^4 \cdot 7}{2^{10} \cdot 3^2}e^7 - \frac{5^6}{2^{13} \cdot 7}e^9 + \frac{5^8 \cdot 11}{2^{18} \cdot 3^3 \cdot 7}e^{11} - \dots\right)\cos 5M \\
& +\left(-\frac{3^3}{2^4 \cdot 5}e^6 + \frac{3^4}{2^2 \cdot 5 \cdot 7}e^8 - \frac{3^6}{2^8 \cdot 7}e^{10} + \frac{3^6}{2^7 \cdot 5 \cdot 7}e^{12} - \dots\right)\cos 6M \\
& +\left(-\frac{7^5}{2^{10} \cdot 3^2 \cdot 5}e^7 + \frac{7^6}{2^{15} \cdot 5}e^9 - \frac{7^9 \cdot 11}{2^{18} \cdot 3^4 \cdot 5 \cdot 7}e^{11} + \dots\right)\cos 7M \\
& +\left(-\frac{2^7}{3^2 \cdot 5 \cdot 7}e^8 + \frac{2^9}{3^4 \cdot 7}e^{10} - \frac{2^{12}}{3^3 \cdot 5^2 \cdot 7}e^{12} + \dots\right)\cos 8M \\
& +\left(-\frac{3^{12}}{2^{15} \cdot 5 \cdot 7}e^9 + \frac{3^{14} \cdot 11}{2^{18} \cdot 5^2 \cdot 7}e^{11} - \dots\right)\cos 9M \\
& +\left(-\frac{5^7}{2^8 \cdot 3^4 \cdot 7}e^{10} + \frac{5^8}{2^7 \cdot 3^3 \cdot 7 \cdot 11}e^{12} - \dots\right)\cos 10M \\
& +\left(-\frac{11^9}{2^{18} \cdot 3^4 \cdot 5^2 \cdot 7}e^{11} + \dots\right)\cos 11M \\
& +\left(-\frac{2 \cdot 3^9}{5^2 \cdot 7 \cdot 11}e^{12} + \dots\right)\cos 12M
\end{aligned}
$$

The Logarithm of the Radius Vector.

The logarithm of the radius vector may be expressed in terms of the eccentricity and mean anomaly, by developing the equation $\frac{r}{a} = 1 - e \cos \mathrm{E}$ by the logarithmic series, thus,

$$\log \frac{r}{a} = -e \cos \mathrm{E} - \frac{e^2}{2} \cos^2 \mathrm{E} - \frac{e^3}{3} \cos^3 \mathrm{E} \quad - \&\mathrm{c}.$$

Transforming this series from powers of cos E to the cosines of multiples of E we obtain the following series, which expresses the logarithm of $\frac{r}{a}$ in terms of the eccentricity and the eccentric anomaly.

$$\begin{aligned}
\log \frac{r}{a} = &-\left(\frac{e^2}{2^2} + \frac{3e^4}{2^5} + \frac{5e^6}{2^5 \cdot 3} + \frac{5 \cdot 7e^8}{2^{10}} + \frac{3^2 \cdot 7e^{10}}{2^9 \cdot 5} + \frac{7 \cdot 11e^{12}}{2^{12}} + \ldots\right) \\
&-\left(e + \frac{e^3}{2^2} + \frac{e^5}{2^3} + \frac{5e^7}{2^6} + \frac{7e^9}{2^7} + \frac{3 \cdot 7e^{11}}{2^9} + \ldots\right) \cos \mathrm{E} \\
&-\left(\frac{e^2}{2^2} + \frac{e^4}{2^3} + \frac{5e^6}{2^6} + \frac{7e^8}{2^7} + \frac{3 \cdot 7e^{10}}{2^9} + \frac{3 \cdot 11e^{12}}{2^{10}} + \ldots\right) \cos 2\mathrm{E} \\
&-\left(\frac{e^3}{2^2 \cdot 3} + \frac{e^5}{2^4} + \frac{3e^7}{2^6} + \frac{7e^9}{2^6 \cdot 3} + \frac{3 \cdot 5e^{11}}{2^9} + \ldots\ldots\right) \cos 3\mathrm{E} \\
&-\left(\frac{e^4}{2^5} + \frac{e^6}{2^5} + \frac{7e^8}{2^8} + \frac{3e^{10}}{2^7} + \frac{3 \cdot 5 \cdot 11e^{12}}{2^{13}} + \ldots\ldots\right) \cos 4\mathrm{E} \\
&-\left(\frac{e^5}{2^4 \cdot 5} + \frac{e^7}{2^6} + \frac{e^9}{2^6} + \frac{3 \cdot 5e^{11}}{2^{10}} + \ldots\ldots\right) \cos 5\mathrm{E} \\
&-\left(\frac{e^6}{2^6 \cdot 3} + \frac{e^8}{2^7} + \frac{3^2e^{10}}{2^{10}} + \frac{5 \cdot 11e^{12}}{2^{11} \cdot 3} + \ldots\ldots\right) \cos 6\mathrm{E} \\
&-\left(\frac{e^7}{2^6 \cdot 7} + \frac{e^9}{2^8} + \frac{5e^{11}}{2^{10}} + \ldots\ldots\right) \cos 7\mathrm{E} \\
&-\left(\frac{e^8}{2^{10}} + \frac{e^{10}}{2^9} + \frac{11e^{12}}{2^{12}} + \ldots\ldots\right) \cos 8\mathrm{E} \\
&-\left(\frac{e^9}{2^8 \cdot 3^2} + \frac{e^{11}}{2^{10}} + \ldots\ldots\right) \cos 9\mathrm{E} \\
&-\left(\frac{e^{10}}{2^{10} \cdot 5} + \frac{e^{12}}{2^{11}} + \ldots\ldots\right) \cos 10\mathrm{E} \\
&-\left(\frac{e^{11}}{2^{10} \cdot 11} + \ldots\ldots\right) \cos 11\mathrm{E} \\
&-\left(\frac{e^{12}}{2^{13} \cdot 3} + \ldots\ldots\right) \cos 12\mathrm{E}
\end{aligned}$$

Substituting the values of cos E, cos 2E, &c., we have

$$
\begin{aligned}
\log \frac{r}{a} = & + \frac{e^2}{2^2} + \frac{e^4}{2^5} + \frac{e^6}{2^5 . 3} + \frac{5e^8}{2^{10}} + \frac{7e^{10}}{2^9 . 5} + \ldots \\
& + \left(- e + \frac{3e^3}{2^3} + \frac{e^5}{2^6} + \frac{127e^7}{2^{10} . 3^2} + \frac{1741e^9}{2^{14} . 3 . 5} + \ldots \right) \cos M \\
& + \left(- \frac{3e^2}{2^2} + \frac{11e^4}{2^3 . 3} - \frac{3e^6}{2^6} + \frac{3^2 e^8}{2^7 . 5} + \frac{349e^{10}}{2^9 . 3^3 . 5} + \ldots \right) \cos 2M \\
& + \left(- \frac{17e^3}{2^3 . 3} + \frac{7 . 11e^5}{2^7} - \frac{743e^7}{2^{10} . 5} + \frac{3539e^9}{2^{13} . 3 . 5} - \ldots \right) \cos 3M \\
& + \left(- \frac{71e^4}{2^5 . 3} + \frac{129e^6}{2^5 . 5} - \frac{387e^8}{2^8 . 5} + \frac{8639e^{10}}{2^7 . 3^3 . 5 . 7} - \ldots \right) \cos 4M \\
& + \left(- \frac{523e^5}{2^7 . 5} + \frac{10039e^7}{2^{10} . 3^2} - \frac{94739e^9}{2^{13} . 3 . 7} + \ldots \right) \cos 5M \\
& + \left(- \frac{899e^6}{2^8 . 3 . 5} + \frac{6617e^8}{2^7 . 5 . 7} - \frac{335711e^{10}}{2^{10} . 5 . 7} + \ldots \right) \cos 6M \\
& + \left(- \frac{355081e^7}{2^{10} . 3^2 . 5 . 7} + \frac{986099e^9}{2^{15} . 3 . 5} - \ldots \right) \cos 7M \\
& + \left(- \frac{47259e^8}{2^{10} . 5 . 7} + \frac{3959051e^{10}}{2^9 . 3^4 . 5 . 7} - \ldots \right) \cos 8M \\
& + \left(- \frac{16541017e^9}{2^{15} . 3^2 . 5 . 7} + \ldots \right) \cos 9M \\
& + \left(- \frac{5719087e^{10}}{2^{10} . 3^4 . 5 . 7} + \ldots \right) \cos 10M
\end{aligned}
$$

Check. Putting $M=0$, it reduces to

$$\log \frac{r}{a} = -\left(e + \frac{e^2}{2} + \frac{e^3}{3} + \frac{e^4}{4} + \ . \ . \ . \ + \frac{e^{10}}{10} + \ . \ . \ .\right)$$

as it should.

When the eccentricity is large, as in the case of the orbits of comets and some of the asteroids, the preceding series for the computation of the eccentric anomaly are not sufficiently converging. In that case the following method may be used:

In the equation $M=E-e \sin E$, let $E=M+x$, that is $x=e \sin E$; then we shall have

$$M = M + x - e \sin (M + x)$$

or

$$x = e \sin M \cos x + e \sin x \cos M$$
$$= e \sin M \left(1 - \frac{x^2}{2} + \ . \ . \ .\right) + e \cos M \left(x - \frac{x^3}{6} + \ . \ . \ .\right)$$

or

$$e \sin M = (1 - e \cos M)x + \frac{e \sin M}{2} x^2 + \frac{e \cos M}{6} x^3 - \ . \ . \ .$$

Reverting this series, we have

$$e \sin E = x$$
$$= \frac{e \sin M}{1 - e \cos M} - \tfrac{1}{2}\left(\frac{e \sin M}{1 - e \cos M}\right)^3 + \ . \ . \ .$$

therefore,

$$E = M + e \sin E$$
$$= M + \frac{e \sin M}{1 - e \cos M} - \tfrac{1}{2}\left(\frac{e \sin M}{1 - e \cos M}\right)^3 + \ . \ . \ .$$

It will generally be sufficient to use only the first two terms of this series which will give the first approximate value of E, which we will denote by E′. With this approximate value compute

$$M' = E' - e \sin E';$$

then, by subtraction, we have

$$M - M' = E - E' - e (\sin E - \sin E')$$
$$= E - E' - e (E - E') \cos E', \text{ approximately;}$$

whence

$$E = E' + \frac{M - M'}{1 - e \cos E'},$$

which is a more accurate value of E.

The last process may be repeated as often as necessary, but unless the first approximation is very far from the truth it will not be necessary to proceed with the computation farther than we have indicated.

This is the same as the method recommended by the late Professor Encke in the *Berliner Astronomisches Jahrbuch*, 1838, and is substantially the same as that given by Gauss in his *Theoria Motus Corporum Cœlestium*, art. ii.

Spottiswoode & Co., Printers, New-street Square, London.

Reprinted from the Monthly Notices of the Royal Astronomical Society, *Vol. XLIV. No.* 2.

THE ORBIT OF THE GREAT COMET (*b*) 1882.

BY

J. MORRISON, M.A., M.D.,

Assistant on the *American Ephemeris and Nautical Almanac*, Washington.

(*Communicated by the Foreign Secretary.*)

Ever since the discovery of this remarkable comet in the early part of September 1882, numerous efforts to determine the elements of its orbit have been made by astronomers in different parts of the world, and in no similar case has there been, we believe, a greater disparity in the results obtained. An orbit like the one under consideration, which is so situated with respect to the orbits of the planets that the effect of planetary perturbations is absolutely inappreciable, must furnish a good example of pure and undisturbed elliptic motion, and therefore we would naturally expect that a system of elements computed from three or four good observations suitably selected would very nearly, if not exactly, represent its motion. In this expectation, however, we are disappointed, for no orbit has hitherto been published which completely satisfies the observations, especially those made after the middle of October or thereabouts. From the time of discovery until about this date the nucleus

presented a tolerably well-defined circular disk which could be as easily observed as that of a planetoid; but soon afterwards extraordinary changes began to take place in the structure of the nucleus, which, from being circular in outline, became oval or elliptical, with from two to four or five bright points or centres of condensation. Thus, from about October 20 until May 26 the length of the nucleus varied from 57″ to about 135″, and the position-angle of its major axis varied from 113° to 255° —which is nearly at right angles to the meridian—hence it is not difficult to see that an error of 30″, or even more, in R.A.. may exist in any observation made about this time, according to the point of the nucleus observed. The error in Declination will be much less than that in R.A. for the reason given above. The following two observations made at the Litchfield Observatory, Clinton, N.Y., and at the Naval Observatory, Washington, are here adduced to show the inconsistency of the results obtained when different points of the nucleus were observed:—

Washington Mean Time.	R.A.	Dec.
	h m s	° ′ ″
1882, Oct. 24·712191	10 6 25·78	−17 6 48·2
24·739461	10 5 52·29	−17 5 6·8

According to the geocentric motion of the comet at that time these observations are inconsistent with each other, and therefore of little or no value in the present investigation. That the discrepancy is due to this cause alone is evident from the fact there were no errors in the position of the comparison star or in the reduction of the observations, which were re-examined by the observers. It is therefore not at all suprising that periods differing very widely from each other have been obtained by different computers, according to the observations employed. Thus, from post-perihelion observations Mr. Chandler obtained periods of 3115 days and 4070 years respectively, and between these extremes we have periods of 269, 652, 712, 793, 843, 997, and 1376 years. The most probable period is between 712 and 793 years, as will appear presently. With the view of comparing the elements obtained from observations made after perihelion passage with those deduced from observations made before and after that epoch, we select the following observations:

	Washington Mean Time.		R.A.	Dec.	Where made.
		d	° ′ ″	° ′ ″	
1	1882, Sept.	8·113405	144 16 52·5	− 0 57 46·4	Windsor
2		8·452672	144 59 51·4	− 0 56 30·0	The Cape
3*		14·348690	161 28 20·0	+ 0 4 12·9	Melbourne
4*		16·368673	170 39 26·3	+ 1 12 4·8	,,
5		16·936700	Ingress on Sun's disk.		The Cape
6*		19·969786	168 34 57·6	− 0 34 28·5	Washington
7	Oct.	8·720457	157 1 37·9	−10 40 21·9	,,

	Washington Mean Time.	R.A.	Dec.	Where made.
	d	° ′ ″	° ′ ″	
8	27·710103	150 31 33·9	−18 14 31·1	Clinton
9*	Nov. 15·740418	141 57 40·8	−24 49 18·9	Washington
10*	24·700923	136 34 3·3	−27 21 26·7	„
11*	Dec. 11·619921	124 5 25·3	−30 16 28·1	„
12	1883, Jan. 13·778694	100 3 54·9	−26 48 39·5	Windsor
13	Feb. 28·304652	87 56 18·0	−15 16 7·1	Washington
14	Apr. 6·097128	89 31 28·9	− 9 4 49·2	Palermo
15	May 6·252250	93 32 12·6	− 6 21 30·3	Cordoba
16	26·257880	97 11 12·6	− 5 30 0·0	„

These observations embrace nearly the entire period of visibility of the comet. No. 7 was made with the equatorial, the comparison star being a well-known one, whose position was subsequently verified by the Transit Circle; No. 8 is marked by the observer as "very good;" in Nos. 9 and 11 the "main point of condensation near the following end of the nucleus was observed," and in 13 "the middle point between the second and third points of the nucleus." No remarks respecting the remaining observations are made by the respective observers.

From Nos. 6, 7, and 13 we have obtained the following system of elements:—

I.

T	1882, Sept. 17·00858 Washington M. T.
ω	69° 37′ 27″·7 Mean Equinox of 1882·0.
☊	346 1 38·8
i	38 0 7·7
log. q	7·8922566
log. e	9·9999660
log. a	1·9992346
P	997·36 years. Motion retrograde.

These elements, although computed with great care, do not represent the motion in the neighbourhood of perihelion. The residuals for the middle place are much greater than what would likely be due to the unavoidable errors of computation and observation. Among quite a number of places computed the smallest residuals obtained were the following:—

		$\Delta\lambda \cos\beta$	$\Delta\beta$
		″	″
3	Sept. 14·34+	+27·21	+17·20
9	Nov. 15·74+	−24·49	+12·53

It is very probable that a slight error in the third place has to some extent vitiated these results. The interval between the first and third observations is 161·3247 days, during which time the comet described an arc of only 16° 12′·1, an amount of motion far too small to permit of an accurate determination of

* Meridian observations.

the elements, especially in orbits like the present one, whose eccentricity is very large.

For the computation of a second system of elements, we will take observations which give a large amount of motion, and therefore select the three meridian observations of September 14, September 19, and November 15; the last being the first meridian observation made after the comet had ceased to be visible in daylight, and had passed from the immediate vicinity of the Sun. At this time, too, the process of disintegration in the nucleus had not advanced very far, and from the remark appended to the observation it is very probable that it is very nearly as accurate as either of the other two about which there is no doubt.

In order to show the degree of accuracy to which the computation has been carried we here give the results of the last two hypotheses. The notation is the same as that used by Gauss in his *Theoria Motus*, with the exception that we have employed u to denote the argument of latitude and ρ the curtate distance. The ratio of the sector to the triangle corresponding to the first and second observations, and also to the first and third, is *negative* and numerically less than unity.

	I.	II.
log. P	9·5492503n	9·5492433n
log. Q	8·4458408n	8·4458447n
$\omega+\sigma$	2° 25′ 49″·788	2° 25′ 50″·016
log. Qc sin ω	0·3691234	0·3691230
z	48° 1′ 1″·943	48° 1′ 1″·936
log. r'	9·3402558	9·3402558
log. ρ'	0·0539758	0·0539758
log. ρ	9·9972618	9·9972631
log. ρ''	0·0734265	0·0734271
log. r	9·3097846	9·3097852
log. r''	0·2226261	0·2226265
☊	346° 0′ 41″·385	346° 0′ 41″·254
i	38 0 7·03	38 0 7·19
$\frac{1}{2}(u''-u)$	164 52 45·703	164 52 47·13
$\frac{1}{2}(u'-u)$	157 55 32·977	157 55 34·27
$\frac{1}{2}(u''-u')$	6 57 12·726	6 57 12·86
log. η	0·1412227	0·1412226
log. η''	9·5876330n	9·5876319n
log. P$_{,}$	9·5492420n	9·5492433n
log. Q$_{,}$	8·4458464n	8·4458450n
x	+0·0000083	0·0000000
y	−0·0000056	−0·0000003

The three values of log p, the semi-parameter, are each equal to 8·1905156, which proves the accuracy of the entire computation up to this point. On completing the computation of the elements from these results, we have the following system :—

II.

T	1882, Sept. 17·010163 Washington M.T.
ω	69° 35′ 30″·483 Mean Equinox of 1882·0.
☊	346 0 41·254
i	38 0 7·19
log. q	7·8895067
log. e	9·9999578
log. a	1·9016841
P	712·10 years. Motion retrograde.

The residuals for the middle place are zero. It now remains to be seen how closely these elements satisfy the above observations.

The residuals are given in the following table, where C—O denotes, as usual, the difference between the *computed* and the *observed* places, and Δ the distance from the earth.

C—O.

	$d\lambda \cos\beta$	$d\beta$	log. Δ
1	− 3″·71	− 0″·28	0·0604274
2	− 0·75	+ 1·02	0·0569198
4	+ 1·60	+ 0·82	9·9900265
7	− 0·16	− 0·44	0·1406414
8	+ 1·01	+ 0·37	0·1662759
10	+ 0·92	− 1·12	0·1766315
11	+ 13·41	+ 6·45	0·1916736
12	− 61·20	+ 37·52	0·2798283
13	− 45·68	+ 3·82	0·4629200
14	− 52·12	− 27·24	0·5880931
15	− 64·07	− 31·20	0·6640691
16	− 54·12	− 20·74	0·7022010

Transit over the Sun's Disk.

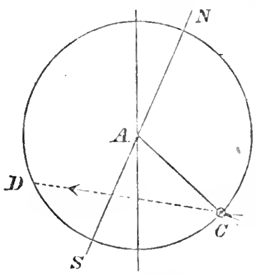
The ingress on the Sun's disk took place on September 17^{d} 4^{h} 50^{m} 58^{s}, Cape mean time, according to Mr. Finlay, and at 4^{h} 50^{m} 52^{s}, according to Dr. Elkin (*Monthly Notices*, November 1882). Adopting the former and applying the correction for aberration, which is −0^{d}·005698, we have for the position of the comet the following results :—

Washington Mean Time.	λ	β	log. Δ
1882, Sept. 16·931001	174° 27′ 19″·83	−0° 13′ 12″·39	9·9952130
Sun's position, ⊙ ...	174 36 19·06	−0 0 0·22	
Difference ...	0 8 59·23	13 12·17	

Distance between the centres	15′ 58″·28
Sun's semi-diameter	15 57·73
Comet's distance from Sun's limb ...	0·55

The diagram shows the position of the comet with respect to the Sun at the time of ingress.

N S is a meridian; C D the apparent path of the comet; N A C the angle of position, 122° 23′·7 from the north point of the Sun's limb towards the west.



Since this second system of elements represents the motion of the comet with great accuracy from the time of discovery till about the middle of November, and from this date until the last observation here recorded, with all the precision that can be expected under the circumstances, we may also give the following facts as tending to show more strongly perhaps the characteristics of this remarkable body.

At the time of ingress on the Sun's disk the comet's true anomaly was 92° 7′ 29″·668, which was described in *one hour and fifty-four minutes.* From the time of the first observation made at Windsor by Mr. Tebbutt until May 26th, an interval of 260·1447 days, the comet described 340° 22′·7 of its orbit, but it will require more than seven centuries to describe the remaining 19° 37′·3. Taking the mean solar parallax at 8″·848, the value adopted in the English and American *Nautical Almanacs*, the comet's perihelion distance was 716200 miles from the Sun's centre, or 285500 miles from the surface.

The velocity at perihelion was 295·36 miles per second, which is only ·0160 miles per second less than that due to a parabola at the same perihelion distance.

When the comet reaches aphelion, it will have plunged down into space a distance equal to 160 radii of the Earth's orbit, or $5\frac{1}{2}$ times the radius of *Neptune's* orbit, at which enormous distance its velocity will be only 75 feet per second.

At the time of Dr. Elkin's first observation on Sept. 8, the apparent diameter of the coma was estimated at from 40″ to 50″, and the strongly condensed cone or nucleus in the centre at from 10″ to 15″. Assuming the means of these measurements,

we easily find the diameter of the coma at that time to have been 22500 miles, and of the nucleus 7600 miles, or a little less than that of the Earth. As the comet approached the Sun, the nucleus contracted, and about half an hour before its disappearance on the solar disk Mr. Finlay found by two micrometric measures the apparent diameter to be 4″, while Dr. Elkin estimated it at some 5″. The former of these gives a diameter of 1770 miles, and the latter 2215 miles, or a little greater than that of the Moon.

The second system of elements here presented proves conclusively that an orbit can be found which satisfies the motion of the body equally well both before and after perihelion; and therefore we may safely conclude that no retardation occurred during the perihelion passage; or, if there was, it is absolutely inappreciable. There are two other systems of elements which differ considerably in the periodic time from our second system, but the agreement of some of the other elements is very remarkable. We refer to those of Prof. Frisby and Dr. Kreutz; the former founded his calculations on the observations of Sept. 19, Oct. 8, and Nov. 24, and the latter on normal places derived from observations extending from Sept. 8 to Nov. 15. The three systems just referred to are here given for comparison.

	Dr. Morrison.	Prof. Frisby.	Dr. Kreutz.
T*	1882, Sept. 17·010163	1882, Sept. 17·0088	1882, Sept. 17·00993
ω†	69° 35′ 30″·48	69° 36′ 12″·79	69° 36′ 1″·50
☊	346 0 41·25	346 1 7·91	346 1 27·20
i	38 0 7·19	38 0 7·84	38 0 19·9
log. *q*	7·8895067	7·8904739	7·8894760
log. *e*	9·9999578	9·9999606	9·9999610
log. *a*	1·9016841	1·9331366	1·9505596
P	712·10 yrs.	793·869 yrs.	843·10 yrs.
	Mean Equinox 1882·0	Mean Equinox 1882·0	Mean Equinox 1882·0

* Washington Mean Time. † Motion retrograde.

This comet is probably identical with the comet of 370 B.C., as already pointed out by Mr. Maxwell Hall. If we divide the interval between 370 B.C. and 1882 A.D. by three we get a period of 751 years, which is very nearly the arithmetic mean between our period and that of Prof. Frisby. This period would give a return in A.D. 1131 or 1132, in both of which years a great comet appeared (*Chinese Annals*).

Spottiswoode & Co., Printers, New-street Square, London.

Reprinted from the MONTHLY NOTICES OF THE ROYAL ASTRONOMICAL SOCIETY, *Vol. XLIV. No.* 7.

THE ORBIT OF PONS' COMET.

BY

J. MORRISON, M.A., M.D.,

Assistant on the American *Ephemeris and Nautical Almanac*,
Washington, D.C.

The Orbit of Pons' Comet. By J. Morrison, M.A., M.D., Assistant on the American *Ephemeris and Nautical Almanac*, Washington, D.C.

The only orbits of this comet which have been published are those of the late Prof. Encke and of Messrs. Schulhof and Bossert: the former deduced from observations extending over a period of only two months in the year 1812, and the latter from observations made in 1883, and subsequently corrected two or three times by equations of condition. Neither Encke's orbit nor that of Schulhof and Bossert represents the motion of the comet with that degree of accuracy which modern investigations demand. In order to obtain an independent determination of the elements, I select the three following observations made at Washington:—

Washington M.T.	Apparent α	Apparent δ
1883 Oct. 10·307425	248° 25′ 56″·1	+56° 51′ 49″·0
Dec. 27·249330	317 34 50·4	+29 31 14·8
1884 Jan. 21·299233	357 36 1·5	−13 31 23·1

The comparison stars were respectively, a *Berliner Jahrbuch*

star; ζ *Cygni*; and a 7th-magnitude star in *Cetus*, as given in the Catalogues of Weisse, Lamont, and Stone.

From these observations I obtain the following system of elements:—

T	1884 Jan. 25·729107, G.M.T.	
ω	199° 12′ 50″·48	Mean Equinox of 1884·0.
☊	254 9 45·82	
i	74 2 0·73	
e	·95414506	
log *q*	9·8896779	
log *a*	1·2282918	

P 69·572 years. Motion Direct.

The residuals for the middle place are zero. It will be noticed that these elements differ considerably from those of Encke and Messrs. Schulhof and Bossert. The following residuals show how closely this system of elements satisfies the observations made from Sept. 5, 1883, to Feb. 5, 1884:

Date. Washington M.T.	C−O cos β dλ	C−O dβ	log Δ	Observation made at
1883 Sept. 5·11 +	− 3″·57	− 3″·72	0·3668189	Milan.
23·32 +	− 3·65	− 5·09	0·3282883	Washington.
Oct. 31·29 +	− 6·14	− 6·59	0·2103271	,,
Nov. 1·33 +	+ 3·73	− 6·90	0·2060407	,,
27·29 +	+ 4·82	+ 7·90	0·0707879	,,
Dec. 6·25 +	+ 10·03	− 1·18	0·0093336	,,
21·24 +	+ 10·60	− 7·20	9·8932833	,,
1884 Jan. 3·23 +	− 3·12	− 2·29	9·8126678	,,
9·30 +	− 0·26	− 1·48	9·8023359	,,
17·30 +	− 0·82	− 4·30	9·8246773	,,
25·29 +	− 1·12	+ 1·10	9·8749017	,,
Feb. 1·28 +	+ 1·78	+ 0·14	9·9258903	,,
5·01 +	− 2·74	+ 1·23	9·9522952	Rome.

Owing to errors in the adopted position of the comparison stars, many of which have not been very accurately determined, and also to the uncertainty always existing in observations of comets in consequence of the ever varying form of the nucleus, these residuals are less than what we may reasonably expect to arise from these causes alone, not to mention the unavoidable errors of computation, even when seven-figure tables are employed, as in the present instance. The formation of equations

of condition for the purpose of correcting these elements, cannot be here applied with any advantage.

The only planets which can have any appreciable influence on the motion of the comet during the time it was in the neighbourhood of perihelion are *Venus* and the Earth. Assuming the elements I have obtained above as the osculating elements for the epoch Jan. 26·0, Greenwich Mean Time, the perturbations of the rectangular co-ordinates, arising from the action of *Venus*, are found to be as follows :—

Greenwich M.T.			δx	δy	δz
1884	Jan.	26·0	−0·0	− 0·0	+0·0
		31·0	−0·1	− 0·3	+0·1
	Feb.	5·0	−0·2	− 1·0	+0·4
		10·0	−0·4	− 2·0	+0·9
		15·0	−0·6	− 3·2	+1·7
		20·0	−0·9	− 4·7	+2·6
		25·0	−1·3	− 6·5	+3·8
	Mar.	1·0	−1·7	− 8·5	+5·0
		6·0	−2·1	−10·8	+6·5
		11·0	−2·6	−13·5	+8·0

which are expressed in units of the *seventh* decimal place. These perturbations do not sensibly affect the inclination, and their influence on the node is less than a second of arc ; they are then almost inappreciable, and, indeed, they scarcely exceed the uncertainty of the seventh decimal place in the final results of extended computations such as the one now under consideration. The perturbations of the rectangular co-ordinates arising from the action of the Earth are even less than those produced by *Venus*, by reason of the Earth being at a greater distance from the comet.

According to my elements, the aphelion distance is 33·056, which exceeds *Neptune's* mean distance from the Sun by about *three* mean radii of the Earth's orbit, and although the inclination is large (74° 2′), the comet will nevertheless be sensibly affected by the action of all the superior planets during the greater part of its period. The amount of this action during the last revolution may, perhaps, be approximately found by comparing my elements with those of Encke, as given in the *Annales de l'Observatoire de Bruxelles*, tom. I. If we refer Encke's elements to the mean equinox and ecliptic of 1884·0, and place them beside mine in order to facilitate comparison, we have the following results :—

	Morrison.	Encke.
T*	1884 Jan. 25·72917	1812, Sept. 15·313553
ω	199° 12′ 50″·48	199° 17′ 5″·89
☊	254 9 45·82	254 1 28·91
i	74 2 0·73	73 56 57·41
e	0·95414506	0·9545412
log q	9·8896779	9·8904995
P	69·572 years	70·684 years

Assuming these values of T to be correct, the actual period of the last revolution was 71·3605 years, and therefore Encke's period has been increased by 247·09 days. It is very probable, however, that Encke's elements are considerably in error, for Messrs. Schulhof and Bossert have deduced from the original observations of 1812 the following elements, which agree tolerably well with mine :—

$\pi -$ ☊ $= \omega$	199° 12′ 32″·5	Mean Equinox, 1880·0
☊	254 6 15·3	
i	74 3 20·4	
e	0·9549960	
log q	9·8893650	

If we add 3′ 20″·92, or four years' precession, to ☊, we get 254° 9′ 36″·2, which is almost identical with my value, while the values of ω and i will not be sensibly affected by this change.

* Greenwich Mean Time.

Spottiswoode & Co., Printers, New-street Square, London.

Reprinted from the MONTHLY NOTICES OF THE ROYAL ASTRONOMICAL SOCIETY, *Vol. XLIV. No.* 8.

THE APPARENT ORBIT OF A SATELLITE OF A SUPERIOR PLANET.

BY

J. MORRISON, M.D., M.A., PH.D.,

Assistant on the American Ephemeris, Washington, D.C.

The Apparent Orbit of a Satellite of a Superior Planet. By J. Morrison, M.D., M.A., Ph.D., Assistant on the American Ephemeris, Washington, D.C.

The apparent orbit of the satellite of a superior planet is the projection of the real orbit on a plane perpendicular to the line joining the Earth and planet, and is, therefore, in general an ellipse. In this paper I shall assume that the elements of the orbit of the satellite are known, and proceed to develop as much of the subject as is necessary for the computation of an ephemeris.

Let

α, δ, Δ be the polar coordinates of the planet as seen from the Earth;

$\alpha', \delta', \Delta'$ the polar coordinates of the satellite as seen from the Earth;

a, d, r the polar coordinates of the satellite as seen from the planet,

referred to the equator and equinox of the epoch for which the ephemeris is to be computed.

The general expressions for the rectangular coordinates in terms of polar coordinates are

$$x = \Delta \cos \delta \cos \alpha$$

$$y = \Delta \cos \delta \sin \alpha$$

$$z = \Delta \sin \delta.$$

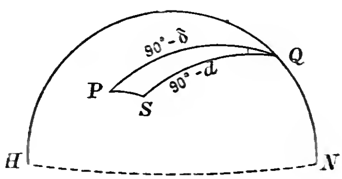
Therefore by equating the rectangular coordinates we have

$$\begin{aligned}\Delta' \cos\delta' \cos\alpha' &= \Delta\cos\delta\cos\alpha + r\cos d\cos a\\ \Delta'\cos\delta'\sin\alpha' &= \Delta\cos\delta\sin\alpha + r\cos d\sin a \qquad \ldots \qquad (1)\\ \Delta'\sin\delta' &= \Delta\sin\delta \qquad + r\sin d.\end{aligned}$$

Multiplying the first of this group by cos α, the second by sin α, and adding the results, then multiplying the first by sin α, the second by cos α, and subtracting, we have

$$\begin{aligned}\Delta'\cos\delta'\cos(\alpha'-\alpha) &= \Delta\cos\delta + r\cos d\cos(a-\alpha)\\ \Delta'\cos\delta'\sin(\alpha'-\alpha) &= r\cos d\sin(a-\alpha) \qquad \ldots \qquad \ldots \qquad (2)\\ \Delta'\sin\delta' &= \Delta\sin\delta + r\sin d.\end{aligned}$$

We have now to consider the spherical triangle formed by the north pole of the equator, the planet, and the satellite; thus let Q be the north pole, P the planet, and S the satellite. In the spherical triangle Q P S we have



$$PQ = 90^\circ - \delta,\ QS = 90^\circ - \delta';$$

and

$$PQS = \alpha' - \alpha.$$

Put

PS = s, the apparent angular distance of the satellite from the planet as seen from the Earth, and

QPS = p, the angle of position of the satellite reckoned from the north toward the east and from 0° to 360°.

By the fundamental formulæ of spherical trigonometry we have

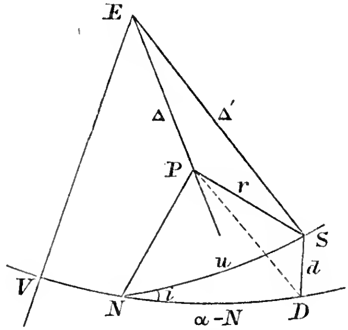
$$\begin{aligned}\sin s\sin p &= \cos\delta'\sin(\alpha'-\alpha)\\ \sin s\cos p &= \cos\delta\sin\delta' - \sin\delta\cos\delta'\cos(\alpha'-\alpha) \qquad \ldots \qquad (3)\\ \cos s &= \sin\delta\sin\delta' + \cos\delta\cos\delta'\cos(\alpha'-\alpha)\end{aligned}$$

Multiplying by Δ' and substituting from (2) this group becomes

$$\begin{aligned}\Delta'\sin s\sin p &= r\cos d\sin(a-\alpha)\\ \Delta'\sin s\cos p &= r\{\cos\delta\sin d - \sin\delta\cos d\cos(a-\alpha)\} \quad \ldots \qquad (4)\\ \Delta'\cos s &= \Delta + r\{\sin\delta\sin d + \cos\delta\cos d\cos(a-\alpha)\},\end{aligned}$$

from which we must eliminate a, d, and Δ'.

Let V D represent an arc of the equator, N S an arc of the orbit of the satellite, E the Earth, P the planet, and S the satellite; draw the arc S D perpendicular to V D, and let E V be the direction of the vernal equinox. Then N will be the



position of the ascending node of the orbit of the satellite on the equator; V D and S D the planetocentric Right Ascension and Declination of the satellite, which we have denoted by a and d respectively.

Put

$$\text{VN} = \text{N, the right ascension of the ascending node;}$$

$$\text{SND} = i\text{, the inclination of the orbit to the plane of the equator;}$$

$$\text{NS} = u\text{, the angular distance of the satellite from the ascending node;}$$

then

$$\text{ND} = a - \text{N}.$$

From the right-angled spherical triangle N S D, whose centre is P, we have

$$\begin{aligned} \cos d \cos (a-\text{N}) &= \cos u \\ \cos d \sin (a-\text{N}) &= \sin u \cos i \quad \ldots \quad \ldots \quad \ldots \quad (5) \\ \sin d &= \sin u \sin i. \end{aligned}$$

Multiplying the first of this group by $\cos(\alpha-\text{N})$, the second by $\sin(\alpha-\text{N})$, and adding the results; also multiplying the first by $\sin(\alpha-\text{N})$ and the second by $\cos(\alpha-\text{N})$, and subtracting, we have

$$\begin{aligned} \cos d \cos (a-\alpha) &= \sin u \cos i \sin (\alpha-\text{N}) + \cos u \cos (\alpha-\text{N}) \\ \cos d \sin (a-\alpha) &= \sin u \cos i \cos (\alpha-\text{N}) - \cos u \sin (\alpha-\text{N}) \\ \sin d &= \sin u \sin i, \end{aligned}$$

which substituted in (4) give, after simple reductions—

$$\Delta' \sin s \sin p = r \{\sin u \cos i \cos (\alpha-\text{N}) - \cos u \sin (\alpha-\text{N})\}$$

$$\Delta' \sin s \cos p = r \{\sin u [\cos \delta \sin i - \sin \delta \cos i \sin (\alpha-\text{N})] \\ - \sin \delta \cos u \cos (\alpha-\text{N})\}$$

$$\Delta' \cos s = \Delta + r \{\sin u [\sin \delta \sin i + \cos \delta \cos i \sin (\alpha-\text{N})] \\ + \cos \delta \cos u \cos (\alpha-\text{N})\},$$

from which Δ' must be eliminated.

In the plane triangle E P S formed by joining the Earth, planet, and satellite, let σ denote the supplement of the angle E P S; then from the diagram (fig. 2) we have

$$\Delta' \sin s = r \sin \sigma$$

and

$$\Delta' \cos s = r \cos \sigma + \Delta \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

which being substituted in the equations of the last group, give after reduction

$$\begin{aligned}
\sin \sigma \sin p &= \sin u \cos i \cos (\alpha - N) - \cos u \sin (\alpha - N) \\
\sin \sigma \cos p &= \sin u [\cos \delta \sin i - \sin \delta \cos i \sin (\alpha - N)] \\
&\qquad - \cos u \sin \delta \cos (\alpha - N) \quad \ldots \quad (7) \\
\cos \sigma &= \sin u [\sin \delta \sin i + \cos \delta \cos i \sin (\alpha - N)] \\
&\qquad + \cos u \cos \delta \cos (\alpha - N).
\end{aligned}$$

These equations are perfectly rigorous, and determine σ and p without ambiguity.

As the angle σ cannot be compared directly with observation, it will be more convenient to express σ in terms of s; thus from (6) we have

$$\tan s = \frac{r \sin \sigma}{r \cos \sigma + \Delta} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

and since s is always very small, we may substitute the arc for the tangent and develope the second member into a series. Hence we have

$$s = \frac{r}{\Delta} \sin \sigma - \frac{1}{2} \frac{r^2}{\Delta^2} \sin 2\sigma + \frac{1}{3} \frac{r^3}{\Delta^3} \sin 3\sigma - \text{ etc.} \quad \ldots \quad (9)$$

Now r is always very small compared with Δ, and therefore the terms containing the square and higher powers of $\frac{r}{\Delta}$ may be neglected without sensibly affecting the result. We shall then have

$$s'' = \frac{r}{\Delta \sin 1''} \sin \sigma \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

and if a'' be the value of $\frac{r}{\sin 1''}$ at distance unit $\frac{a''}{\Delta}$ will be its value at distance Δ; therefore the last equation may be written

$$s'' = \frac{a''}{\Delta} \sin \sigma,$$

whence

$$\sin \sigma = \frac{\Delta}{a''} \cdot s'' \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11$$

which, substituted in the first two equations of (7), gives

$$s'' \sin p = \frac{a''}{\Delta}\left\{\sin u \cos i \cos(\alpha - \text{N}) - \cos u \sin(\alpha - \text{N})\right\} \quad \ldots \quad \ldots \quad (12)$$

$$s'' \cos p = \frac{a''}{\Delta}\left\{\sin u\,[\cos\delta \sin i - \sin\delta \cos i \sin(\alpha - \text{N})] - \cos u \sin\delta \cos(\alpha - \text{N})\right\}$$

The error committed in s'' by omitting the second term of (9) is

$$\tfrac{1}{2}\,\frac{r^2}{\Delta^2}\,\frac{\sin 2\sigma}{\sin 1''},$$

which is a maximum when

$$\sigma = 45^\circ \text{ or } 135^\circ.$$

Put

$$\tfrac{1}{2}\,\frac{r^2}{\Delta^2}\,\frac{1}{\sin 1''} = \eta\,;$$

then

$$\frac{r}{\Delta} = \sqrt{\eta \sin 2''}\,;$$

but $\frac{r}{\Delta}$ is the tangent of the angle of greatest elongation as seen from the earth; hence, putting

$$\frac{r}{\Delta} = \tan\epsilon,$$

we have

$$\eta = \frac{\tan^2\epsilon}{\sin 2''},$$

when

$$\begin{array}{ll} \epsilon = 0'\ 50'', & \eta = 0''{\cdot}006 \\ \epsilon = 1\ \ 50, & \eta = 0{\cdot}029 \\ \epsilon = 3\ \ 35, & \eta = 0{\cdot}112 \\ \epsilon = 5\ \ 46, & \eta = 0{\cdot}290 \\ \epsilon = 9\ \ 45, & \eta = 0{\cdot}829 \end{array}$$

Thus we see that for the satellites of *Mars*, *Uranus*, *Neptune*, the three inner satellites of *Jupiter*, and the five inner satellites of *Saturn* the error committed in s'' is less than $0''{\cdot}3$, and for *Callisto*, the fourth satellite of *Jupiter*, which has the largest angle of elongation, the error can never exceed $0''{\cdot}83$. In order to adopt (7) and (12) to logarithmic computation put

$$\begin{aligned} \sin f \cos \text{F} &= \cos i \cos(\alpha - \text{N}) \\ \sin f \sin \text{F} &= -\sin(\alpha - \text{N}) \qquad \ldots \quad \ldots \quad \ldots \quad (13) \\ \cos f &= -\sin i \cos(\alpha - \text{N}) \end{aligned}$$

$$\begin{aligned} \sin g \cos \text{G} &= \cos\delta \sin i - \sin\delta \cos i \sin(\alpha - \text{N}) \\ \sin g \sin \text{G} &= -\sin\delta \cos(\alpha - \text{N}) \qquad \ldots \quad \ldots \quad (14) \\ \cos g &= \cos\delta \cos i + \sin\delta \sin i \sin(\alpha - \text{N}) \end{aligned}$$

$$\begin{aligned}\sin h \cos \mathrm{H} &= \sin \delta \sin i + \cos \delta \cos i \sin (\alpha - \aleph)\\ \sin h \sin \mathrm{H} &= \cos \delta \cos (\alpha - \aleph) \quad \ldots \quad \ldots \quad \ldots \quad (15)\\ \cos h &= \sin \delta \cos i - \cos \delta \sin i \sin (\alpha - \aleph).\end{aligned}$$

Hence (7) and (12) become

$$\begin{aligned}\sin \sigma \sin p &= \sin f \sin (\mathrm{F} + u)\\ \sin \sigma \cos p &= \sin g \sin (\mathrm{G} + u) \quad \ldots \quad (16)\\ \cos \sigma &= \sin h \sin (\mathrm{H} + u)\end{aligned}$$

and

$$\begin{aligned}s'' \sin p &= \frac{a''}{\Delta} \sin f \sin (\mathrm{F} + u)\\ s'' \cos p &= \frac{a''}{\Delta} \sin g \sin (\mathrm{G} + u) \quad \ldots \quad \ldots \quad (17)\end{aligned}$$

The last group, which is known as Bessel's formulæ, gives s'' and p for any given date, and also enables us to compare the computed place with that obtained by observation. When s and p are to be computed for a series of dates it will be most convenient to compute the values of the auxiliary angles, f, F, g, and G, for the mean noon of several consecutive days, and then interpolate for any intermediate date. The auxiliaries h and H will not be required unless we also want the value of σ.

The following relations among the auxiliary angles may be here noticed, since they will be required presently in the reduction of the differential equations.

By combining the values of $\sin f \sin \mathrm{F}$, $\sin g \cos \mathrm{G}$, $\sin f \cos \mathrm{F}$, and $\sin g \sin \mathrm{G}$, we have

$$\begin{aligned}\sin f \sin g \sin \mathrm{F} \cos \mathrm{G} &= \sin^2 (\alpha - \aleph) \cos i \sin \delta - \sin (\alpha - \aleph) \sin i \cos \delta\\ \sin f \sin g \cos \mathrm{F} \sin \mathrm{G} &= -\cos^2 (\alpha - \aleph) \cos i \sin \delta;\end{aligned}$$

therefore by subtraction we have

$$\begin{aligned}\sin f \sin g \sin (\mathrm{F} - \mathrm{G}) &= \cos i \sin \delta - \sin (\alpha - \aleph) \sin i \cos \delta\\ &= \cos h \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)\end{aligned}$$

In a similar manner we find

$$\begin{aligned}\sin g \sin h \sin (\mathrm{G} - \mathrm{H}) &= \cos f \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)\\ \sin f \sin h \sin (\mathrm{H} - \mathrm{F}) &= \cos g \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)\end{aligned}$$

Again, multiplying the first and second of (15) by $\cos u$ and $\sin u$ respectively, we have, after obvious reductions—

$$\begin{aligned}&\sin h \cos (\mathrm{H} + u)\\ &\quad = [\sin i \sin \delta + \cos i \cos \delta \sin (\alpha - \aleph)] \cos u - \cos \delta \cos (\alpha - \aleph) \sin u;\end{aligned}$$

also, multiplying the second members of the first and second of (16) by the third of (14) and (13) respectively, we have, by the aid of (7)—

$$\begin{aligned}&\cos f \sin g \sin (\mathrm{G} + u) - \sin f \cos g \sin (\mathrm{F} + u) = [\sin i \sin \delta\\ &\quad + \cos i \cos \delta \sin (\alpha - \aleph)] \cos u - \cos \delta \cos (\alpha - \aleph) \sin u.\end{aligned}$$

Comparing this with the preceding equation, we have the following relation:

$$\sin h \cos (\mathrm{H}+u) = \cos f \sin g \sin (\mathrm{G}+u) - \sin f \cos g \sin (\mathrm{F}+u) \quad (21)$$

To Correct the Elements.

When the computed places do not agree with those obtained by observation it will be necessary to correct the elements by forming equations of condition, which will be of the form

$$\Delta p = \frac{dp}{du}\Delta u + \frac{dp}{d\mathrm{N}}\Delta \mathrm{N} + \frac{dp}{di}\Delta i$$

$$\Delta s = \frac{ds}{du}\Delta u + \frac{ds}{d\mathrm{N}}\Delta \mathrm{N} + \frac{ds}{di}\Delta i + \frac{ds}{da}\Delta a,$$

where we consider for the present the elements of a circular orbit.

Differentiating the first of (7) we get (α and δ being constants)

$$\begin{aligned} \sin \sigma \cos p dp + \cos \sigma \sin p d\sigma = & \ [\sin (\alpha - \mathrm{N}) \sin u + \cos (\alpha - \mathrm{N}) \cos i \cos u]\, du \\ & + [\cos (\alpha - \mathrm{N}) \cos u + \sin (\alpha - \mathrm{N}) \cos i \sin u] d\mathrm{N} \\ & - \cos (\alpha - \mathrm{N}) \sin i \sin u di \\ = & \ \sin f \cos (\mathrm{F}+u) du \\ & + [\cos (\alpha - \mathrm{N}) \cos u \\ & \qquad + \sin (\alpha - \mathrm{N}) \cos i \sin u] d\mathrm{N} \quad (a) \\ & + \cos f \sin u di \end{aligned}$$

Differentiating the second of (7) gives

$$\begin{aligned} -\sin \sigma \sin p dp + \cos \sigma \cos p d\sigma = & \ [\cos (\alpha - \mathrm{N}) \sin \delta \sin u - \sin (\alpha - \mathrm{N}) \\ & \qquad \cos i \sin \delta \cos u + \sin i \cos \delta \cos u] du \\ & + [\cos (\alpha - \mathrm{N}) \cos i \sin \delta \sin u \\ & \qquad - \sin (\alpha - \mathrm{N}) \sin \delta \cos u] d\mathrm{N} \\ & + [\sin (\alpha - \mathrm{N}) \sin i \sin \delta \sin u \\ & \qquad + \cos i \cos \delta \sin u] di \\ = & \ \sin g \cos (\mathrm{G}+u) du \\ & + \sin \delta \sin f \sin (\mathrm{F}+u) d\mathrm{N} \quad \ldots \quad (b) \\ & + \cos g \sin u di \end{aligned}$$

Multiplying (a) by the second of (16), and (b) by the first of (16), and subtracting the latter result from the former, we have, after reducing by the aid of the relations (18–21)—

$$\begin{aligned} \sin^2 \sigma dp = & -\cos h du \\ & + [\sin i \sin h \sin (\mathrm{H}+u) - \sin \delta] d\mathrm{N} \quad \ldots \quad (22) \\ & + \sin h \cos (\mathrm{H}+u) \sin u di. \end{aligned}$$

Differentiating the third of (7), we find after similar reductions

$$\begin{aligned}\sin\sigma\, d\sigma = & -\sin h \cos(\mathrm{H}+u)du \\ & +\cos\delta \sin f \sin(\mathrm{F}+u)d\mathrm{N} \quad \ldots \quad \ldots \quad (23) \\ & -\cos h \sin u di.\end{aligned}$$

From the plane triangle E P S (fig. 2) we have

$$\Delta' = \Delta \cos s + r\cos(\sigma - s)$$

and

$$\sin s = \frac{r}{\Delta}\sin(\sigma - s).$$

Differentiating the latter and reducing by the former we have

$$\frac{\sin\sigma}{\sin s}ds = \cos(\sigma-s)d\sigma + \sin(\sigma-s)\frac{dr}{r} \quad \ldots \quad \ldots \quad (24)$$

whence

$$d\sigma = \frac{\sin\sigma}{\sin s\cos(\sigma-s)}ds - \tan(\sigma-s)\frac{da}{a},$$

where we write a for r in the circular orbit.

This being substituted in (23) gives, after putting for brevity

$$\sin s\cos(\sigma - s) = n,$$

$$\begin{aligned}\sin^2\sigma\, ds = & -n\sin h\cos(\mathrm{H}+u)du \\ & +n\cos\delta\sin f\sin(\mathrm{F}+u)d\mathrm{N} \\ & -n\cos h\sin u di \\ & +\frac{n}{a}\sin\sigma\tan(\sigma-s)da \quad \ldots \quad \ldots \quad (25)\end{aligned}$$

Therefore the partial differential coefficients of h and s with respect to the elements of the circular orbit are

$$\begin{aligned}\sin^2\sigma\frac{dp}{du} &= -\cos h \\ \sin^2\sigma\frac{dp}{d\mathrm{N}} &= \sin i\sin h\sin(\mathrm{H}+u)\sin u - \sin\delta \quad \ldots \quad (26) \\ \sin^2\sigma\frac{dp}{di} &= \sin h\cos(\mathrm{H}+u)\sin u,\end{aligned}$$

and

$$\begin{aligned}\sin^2\sigma\frac{ds}{du} &= -n\sin h\cos(\mathrm{H}+u) \\ \sin^2\sigma\frac{ds}{d\mathrm{N}} &= n\cos\delta\sin f\sin(\mathrm{F}+u) \\ \sin^2\sigma\frac{ds}{di} &= -n\cos h\sin u \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27) \\ \sin^2\sigma\frac{ds}{da} &= \frac{n}{a}\sin\sigma\tan(\sigma-s),\end{aligned}$$

where s and a are to be expressed in seconds of arc at distance Δ.

If, however, we take the eccentricity of the orbit into consideration, we must express du and da or dr in terms of the remaining elements of the elliptic orbit. Let

m_0 denote the mean anomaly at any given epoch T_0,

μ the mean motion in a solar day,

ϕ the angle of eccentricity such that $\sin\phi = e$,

E the eccentric anomaly,

v the true anomaly,

w the angular distance from the ascending node to the pericentre,

and

m the mean anomaly for any subsequent date t;

then by the theory of elliptic motion we have

$$\begin{aligned} m &= m_0 + (t - T_0)\mu \\ &= E - \sin\phi \sin E \end{aligned}$$

$$\begin{aligned} E &= m + (\sin\phi - \tfrac{1}{8}\sin^3\phi + \ \ldots\ )\sin m \\ &\quad + \tfrac{1}{2}\sin^2\phi \sin 2m - \ \ldots\ldots \end{aligned}$$

$$\begin{aligned} \cos E &= -\tfrac{1}{2}\sin\phi + (1 - \tfrac{3}{8}\sin^2\phi + \ \ldots)\cos m \\ &\quad + \tfrac{1}{2}\sin\phi \cos 2m - \ \ldots\ldots \end{aligned}$$

$$\begin{aligned} r &= a(1 - \sin\phi \cos E) \\ &= a(1 - \sin\phi\cos m + \tfrac{1}{2}\sin^2\phi - \ \ldots) \quad \ldots \quad (28) \end{aligned}$$

$$v = m + 2\sin\phi\sin m + \tfrac{5}{4}\sin^2\phi\sin 2m + \ \ldots$$

and also

$$\begin{aligned} u &= w + v \\ &= w + m + 2\sin\phi\sin m + \ \ldots\ldots \quad \ldots \quad (29) \end{aligned}$$

Since the eccentricity of the orbits of all the satellites of the superior planets is not only small but also very uncertain, we may neglect the square and higher powers of $\sin\phi$ without sensibly affecting the accuracy of our results.

Differentiating (29) gives

$$du = dw + dm + 2\sin\phi\cos m\,dm + 2\sin m\cos\phi\,d\phi\,;$$

and expressing the radius vector of the satellite in terms of the semi-major axis (a) of its orbit by putting

$$\frac{r}{a} = \rho,$$

we shall have by differentiating this last equation

$$\frac{dr}{r} = \frac{da}{a} + \frac{d\rho}{\rho},$$

where r and a are expressed in the same unit as Δ. Therefore from (28) we have

$$d\rho = d\left(\frac{r}{a}\right) = \sin\phi \sin m dm - \cos m \cos\phi d\phi;$$

hence

$$\frac{dr}{r} = \frac{da}{a} + \frac{\sin\phi \sin m}{\rho} dm - \frac{\cos m \cos\phi}{\rho} d\phi.$$

Therefore by substituting these values of du and $\frac{dr}{r}$ in (22), (24), and (25) we have for the partial differential coefficients of p and s with respect to the elements of the elliptic orbit the following results:—

$$\begin{aligned}
\sin^2\sigma\frac{dp}{dw} &= -\cos h \\
\sin^2\sigma\frac{dp}{dm} &= -(1 + 2\sin\phi\cos m)\cos h \\
\sin^2\sigma\frac{dp}{d\phi} &= -2\sin m\cos\phi\cos h \quad \ldots \quad \ldots \quad \ldots \quad (30) \\
\sin^2\sigma\frac{dp}{dN} &= \sin i\sin h\sin(H+u)\sin u - \sin\delta \\
\sin^2\sigma\frac{dp}{di} &= \sin h\cos(H+u)\sin u
\end{aligned}$$

and

$$\begin{aligned}
\sin^2\sigma\frac{ds}{dw} &= -n\sin h\cos(H+u) \\
\sin^2\sigma\frac{ds}{dm} &= -(1 + 2\sin\phi\cos m)n\sin h\cos(H+u) \\
&\quad + \frac{n}{\rho}\sin\sigma\tan(\sigma - s)\sin\phi\sin m \\
\sin^2\sigma\frac{ds}{d\phi} &= -[2n\sin m\sin h\cos(H+u) \\
&\quad + \frac{n}{\rho}\sin\sigma\tan(\sigma - s)\cos m]\cos\phi \\
\sin^2\sigma\frac{ds}{dN} &= n\cos\delta\sin f\sin(F+u) \ldots \quad \ldots \quad \ldots \quad (31) \\
\sin^2\sigma\frac{ds}{di} &= -n\cos h\sin u \\
\sin^2\sigma\frac{ds}{da} &= \frac{n}{a}\sin\sigma\tan(\sigma - s),
\end{aligned}$$

in which a is to be expressed in seconds of arc at distance Δ.

To Compute the Times of greatest Eastern and Western Elongation.

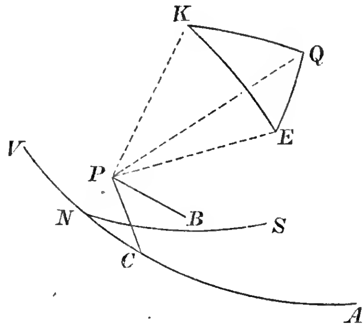
Let us conceive three planes to be passed as follows:—

The first through the centre of the planet and the poles of the equator and the orbit of the satellite;

The second through the centre of the planet, the pole of the orbit of the satellite, and the centre of the Earth;

The third through the centre of the planet, the pole of the equator, and the centre of the Earth.

These three planes will determine on the celestial sphere a spherical triangle having its centre at the centre of the planet.



Let V A represent an arc of the equator, N S an arc of the orbit of the satellite, Q the pole of the equator, K the pole of the orbit of the satellite, N the position of the ascending node, P the planet, E the Earth.

Join P E, P K, P Q, and in the plane K P E draw P B perpendicular to K P, and P C perpendicular to P E; then P B will lie in the plane of the orbit, and P C in the plane of the apparent orbit, which is the projection of the real orbit on the plane passing through P, and perpendicular to the line P E joining the Earth and planet.

Put

θ = the angle EPB,

q = the angle of position of the minor axis of the apparent orbit,

$p_0 = q \pm 90°$, the angle of position of the major axis,

u_0 = the angular distance of the end of the major axis from the ascending node measured on the orbit of the satellite,

a and b = the semi major and minor axes respectively of the apparent orbit;

then in the spherical triangle E K Q we shall have

KQ = i, the inclination of the orbit of the satellite,

EQ = $90° + \delta$, the polar distance of the Earth as seen from the planet, and

EQK = the difference between the Right Ascension of the Earth and the Right Ascension of the pole of the orbit of the satellite

$= (\alpha - 180°) - (N - 90°)$

$= \alpha - N - 90°$;

hence the remaining three parts of the triangle can be found.

Since the planes P K E, P K Q pass through the pole of the orbit, they are perpendicular to its plane, and the former intersects the orbit in those points which, as seen from the Earth, are nearest the planet, or, in other words, the intersection of the plane P K E with the apparent orbit, determining the minor axis, while the plane P K Q, being perpendicular to the planes of the equator and of the orbit of the satellite, intersects the latter at a point 90° distant from the node. The angle E K Q, being the inclination of these two planes, is measured by the arc of the orbit intercepted between these two points of intersection.

The side K Q will intersect the orbit in a point whose position is

$$90^\circ + N;$$

and if we denote the position of the intersection of the side K E with the orbit by β we shall have

$$\text{angle EKQ} = 90^\circ + N - \beta.$$

But since the plane P K E intersects the apparent orbit in its minor axis, the position of that point of the orbit at which the satellite, as seen from the Earth, will be at its greatest eastern elongation will be

$$\beta - 90^\circ,$$

and therefore the distance of this point from the node will be

$$\beta - 90^\circ - N.$$

Hence we have

$$u_\circ = \beta - 90^\circ - N.$$

But we have shown that

$$\text{angle EKQ} = -\beta + 90^\circ + N;$$

therefore the angle E K Q $= -u_\circ$.

The angle K E Q is the angle of position of the minor axis of the apparent orbit; hence

$$\text{angle KEQ} = q = p_\circ - 90^\circ.$$

The side K E is the angular distance of the Earth from the pole of the orbit, as seen from the planet, and is equal to the angle B P C; therefore we have

$$\text{KE} = 90^\circ - \theta,$$

where θ is always to be taken between $+90^\circ$ and -90°. When θ is *positive* the Earth will be on the *north* side of the plane of the orbit and the position angles will *increase*, and when θ is *negative* the Earth will be on the *south* side and the position angles will *decrease*.

The fundamental formulæ of spherical trigonometry being applied to the triangle E K Q, we have

$$\begin{aligned}
\cos \mathrm{EK} &= \cos \mathrm{KQ} \cos \mathrm{EQ} + \sin \mathrm{KQ} \sin \mathrm{EQ} \cos \mathrm{KQE} \\
\sin \mathrm{EK} \cos \mathrm{EKQ} &= \sin \mathrm{KQ} \cos \mathrm{EQ} - \cos \mathrm{KQ} \sin \mathrm{EQ} \cos \mathrm{KQE} \\
\sin \mathrm{EK} \sin \mathrm{EKQ} &= \sin \mathrm{EQ} \sin \mathrm{KQE} \\
\sin \mathrm{EK} \cos \mathrm{KEQ} &= \sin \mathrm{EQ} \cos \mathrm{KQ} - \cos \mathrm{EQ} \sin \mathrm{KQ} \cos \mathrm{KQE} \\
\sin \mathrm{EK} \sin \mathrm{KEQ} &= \sin \mathrm{KQ} \sin \mathrm{KQE}.
\end{aligned}$$

But we have shown that

$$\begin{aligned}
\mathrm{KQ} &= i, \\
\mathrm{EQ} &= 90^\circ + \delta, \\
\mathrm{KQE} &= \alpha - \mathrm{N} - 90^\circ, \text{ which are known;}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{EK} &= 90^\circ - \theta, \\
\mathrm{EKQ} &= -u_\circ, \\
\mathrm{KEQ} &= p_\circ - 90^\circ, \text{ which are to be found.}
\end{aligned}$$

Substituting these values of the sides and angles in the preceding formulæ, we have

$$\begin{aligned}
\sin \theta &= -\cos i \sin \delta + \sin i \cos \delta \sin (\alpha - \mathrm{N}) \\
\cos \theta \cos u_\circ &= -\sin i \sin \delta - \cos i \cos \delta \sin (\alpha - \mathrm{N}) \\
\cos \theta \sin u_\circ &= \cos \delta \cos (\alpha - \mathrm{N}) \quad \ldots \quad (32) \\
\cos \theta \sin p_\circ &= \cos i \cos \delta + \sin i \sin \delta \sin (\alpha - \mathrm{N}) \\
\cos \theta \cos p_\circ &= \sin i \cos (\alpha - \mathrm{N})
\end{aligned}$$

for the determination of θ, $u_\circ$, and $p_\circ$. The agreement of the two values of $\cos \theta$ and the one value of $\sin \theta$ will serve to check the accuracy of the computation. It now remains to transform these equations for logarithmic calculation.

If we put

$$\begin{aligned}
c \sin \mathrm{C} &= \cos i \\
c \cos \mathrm{C} &= \sin i \sin (\alpha - \mathrm{N}),
\end{aligned}$$

$$\begin{aligned}
c' \sin \mathrm{C}' &= \sin i \\
c' \cos \mathrm{C}' &= -\cos i \sin (\alpha - \mathrm{N}),
\end{aligned}$$

and remembering that for the greatest western elongation $-u_\circ$ becomes $-u_\circ + 180^\circ$, and $p_\circ$ becomes $p_\circ + 180^\circ$, we shall have

$$\begin{aligned}
\sin \theta &= c \cos (\mathrm{C} + \delta) \\
\cos \theta \cos u_\circ &= \pm c' \cos (\mathrm{C}' + \delta) \\
\cos \theta \sin u_\circ &= \pm \cos \delta \cos (\alpha - \mathrm{N}) \quad \ldots \quad \ldots \quad (33) \\
\cos \theta \sin p_\circ &= \pm c \sin (\mathrm{C} + \delta) \\
\cos \theta \cos p_\circ &= \pm \sin i \cos (\alpha - \mathrm{N}),
\end{aligned}$$

in which the upper sign is to be used for the greatest eastern, and the lower sign for the greatest western elongation.

Now if u denote the distance of the satellite from the ascending node at the date T, and if t denote the time of greatest elongation, we shall have

$$(t - \mathrm{T})\,\mu = u_0 - u,$$

whence

$$t = \mathrm{T} + \frac{u_0 - u}{\mu} \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

We also have

$$\begin{aligned} b &= a \cos \mathrm{KE} \\ &= a \sin \theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (35) \end{aligned}$$

which determines the semi-minor axis of the apparent orbit.

Spottiswoode & Co., Printers, New-street Square, London.

Reprinted from the Monthly Notices of the Royal Astronomical Society, *Vol. XLV. No.* 1.

THE ORBIT OF BARNARD'S COMET, 1884.

BY

J. MORRISON, M.D., Ph.D.

Assistant on the American Ephemeris, Washington, D.C.

The Orbit of Barnard's Comet, 1884. By J. Morrison, M.D., Ph.D., Assistant on the American Ephemeris, Washington, D.C.

From the numerous observations of this comet, already published, I select for the purpose of computing elliptic elements the following observations made at Cambridge, Mass., Nice, and Washington respectively:—

Washington M.T.	α	δ
	° ′ ″	° ′ ″
July 25·371990	242 4 4·5	−37 15 32·0
Aug. 24·124442	267 10 42·3	−35 15 30·4
Sept. 23·339919	297 55 9·3	−26 49 50·6

From these positions I obtain by the usual methods the following system of elliptic elements:—

Epoch	1884, Sept. 24·5 Wash. M.T.	
M	7° 13′ 19″·52	Mean Equinox of 1884·0.
ω	300 57 44·43	
☊	5 11 23·56	
i	5 27 18·94	
ϕ	35 37 2·50	
$\log q$	0·1069968	
$\log a$	0·4862043	
$\log \mu$	2·8207001	
P	1958·41 days.	

The residuals for the middle place are zero, and for the following dates the difference between the computed and observed places is as small as can be expected, considering the great difficulty in observing accurately so faint an object:—

Wash. M.T.	C−O. $d\lambda \cos\beta$.	C−O. $d\beta$.	log. Δ.
1884, Aug. 12·361652	+ 6″·52	− 3″·21	9·65375
Sept. 15·408454	+ 9·85	− 8·09	9·74018
Oct. 11·318804	+30·91	−10·62	9·86241
14·365442	+21·56	+15·71	9·87849

Spottiswoode & Co. Printers, New-street Square, London.

Reprinted from the Monthly Notices of the Royal Astronomical Society,
Vol. XLVII. No. 7.

THE ORBITS OF COMETS FABRY AND BARNARD-HARTWIG.

EPHEMERIDES OF THE SATELLITES OF MARS DURING THE OPPOSITIONS OF 1888 AND 1890.

BY

J. MORRISON, M.D., Ph.D.

Assistant on the American Ephemeris, and Professor of Chemistry
in the National University, Washington.

The Orbits of Comets Fabry and Barnard-Hartwig. By J. Morrison, M.D., Ph.D., Assistant on the American Ephemeris, and Professor of Chemistry in the National University, Washington.

Comet Fabry.

The observations upon which the following hyperbolic elements of this comet are founded are as follow :—

Greenwich M.T.	Appar. α	Appar. δ
d	h m s	° ′ ″
1885, Dec. 7·536032	0 24 46·01	+20 52 34·9
1886, Mar. 7·316539	23 19 42·92	+31 16 44·8*
„ April 1·339713	23 18 30·86	+38 37 57·0
„ June 6·953118	8 47 52·15	−40 38 0·5

The first is a meridian observation made at Ann Arbor; the second results from extra meridian observations made at Greenwich and Paris, the comparison star being the same at both places; the third was made at Bothkamp and was obtained from *Astronomische Nachrichten*, No. 2703; and the fourth was made at Sydney, the comet being at the time "extremely faint, but in a good position for observation with Cape Cat. (1880) 4707," fifty comparisons having been made. (*Monthly Notices*, vol. xlvi. p. 496.) These observations were corrected for aberration and parallax by means of approximate parabolic elements.

* Geocentric.

T	1886, April $5^{d}\cdot9520$ Greenwich M.T.
ω	$126^{\circ}\ 34'\ 49''\cdot215$
☊	36 22 11·454 Mean Equinox, 1886·0
i	82 37 6·012
e	1·00047857
log a	3·1278354
log q	9·8077809

The two middle places are well represented.

The interval between the extreme observations is 181·42 days, during which the comet described 208° 9′ of its orbit.

Comet Barnard-Hartwig. 1886.

From the following observations the first and third of which were made at Washington, D.C., and the second at Kiel (*Astronomische Nachrichten*, 2753), an elliptic orbit of long period is obtained, the elements of which are given below :—

Greenwich M.T. d	Appar. α. h m s	Appar. δ. ° ′ ″
1886, Oct. 7·918987	10 42 9·32	+ 1 22 6·3
„ „ 29·708660	11 39 22·82	+ 5 49 19·4
„ Dec. 2·976975	15 32 3·91	+17 58 55·8

By means of approximate parabolic elements computed from a shorter interval, the corrections for aberration and parallax were obtained and applied.

T	1886, Dec. $16^{d}\cdot514158$ Greenwich M.T.
ω	$86^{\circ}\ 21'\ 58''\cdot570$
☊	137 21 36·163 Mean Equinox, 1886·0
i	78 22 25·525
e	0·99872521
log a	2·7162151
log q	9·8216538

Motion retrograde. The middle place is exactly represented. These elements give a period of 11,866 years, which is of course very uncertain, since the interval between the extreme observations is far too short to determine this element accurately in an orbit such as this is.

The formulæ for the equatorial rectangular coordinates are—

$$x=[9\cdot8740086]\,r\sin(v+\ \ 6^{\circ}\ 52'\ 45''\cdot171)$$
$$y=[9\cdot8268980]\,r\sin(v+198\ \ 35\ \ \ 9\cdot651)$$
$$z=[9\cdot9977333]\,r\sin(v+102\ \ \ 5\ \ 28\cdot657)$$

Ephemerides of the Satellites of Mars during the Oppositions of 1888 and 1890. By J. Morrison, M.D., M.A., Ph.D., Assistant on the American Ephemeris, and Professor of Chemistry, National University, Washington.

These ephemerides have been computed from the following elements of the orbits of the satellites, referred to the equator and equinox of the respective epochs:—

Phobos.

Epochs	1888, April 11·0	1890, May 27·0 Greenwich M.T.
Period	0·3189113 d (mean solar)	
μ	1128°·8405	
a	12″·953 (at distance unity)	
i	36° 44′·6	36° 44′·0
N	47 18·2	47 19·0
u	17 40	117 53·6

Deimos.

Epochs	1888, April 11·0	1890, May 27·0 Greenwich M.T.
Period	1·262435 d (mean solar)	
μ	285°·16322	
a	32″·354 (at distance unity)	
i	35° 36′	35° 35′·5
N	48 10·66	48 11·5
u	225 41	172 21·5

Greenwich Mean Time of Greatest Elongation.

Phobos.

1888. d	G.M.T. h m	p °	a ″	b ″	1888. d	G.M.T. h m	p °	a ″	b ″
Mar. 21	7 53·5 W	306·4	19·1	6·1	Mar. 30	6 8·5 W			
22	10 40·4 E				31	8 55·3 E			
23	13 27·3 W				Apr. 1	11 42·1 W	305·8	20·5	6·8
24	16 14·2 E				2	14 29·0 E			
25	19 1·1 W				3	17 15·8 W			
26	21 48·0 E	126·2	19·6	6·4	4	20 2·6 E			
28	0 34·8 W				5	22 49·4 W			
29	3 21·7 E				7	1 36·1 E	125·3	21·0	7·3

1888. G.M.T. d h m	p °	a ″	b ″	1888. G.M.T. d h m	p °	a ″	b ″
Apr. 8 4 22·8 W				Apr. 24 22 4·4 E			
9 7 9·6 E				26 0 51·3 W			
10 9 56·3 W				27 3 38·2 E			
11 12 43·1 E				28 6 25·1 W			
12 15 29·8 W	304·9	21·3	7·6	29 9 11·9 E	123·3	21·0	8·0
13 18 16·6 E				30 11 58·8 W			
14 21 3·3 W				May 1 14 45·8 E			
15 23 50·1 E				2 17 32·7 W			
17 2 36·9 W				3 20 19·7 E			
18 5 23·7 E	124·4	21·4	7·8	4 23 6·7 W	302·7	20·5	8·1
19 8 10·5 W				6 1 53·8 E			
20 10 57·2 E				7 4 40·9 W			
21 13 44·0 W				8 7 28·0 E			
22 16 30·8 E				9 10 17·1 W			
23 19 17·6 W	303·7	21·2	8·0	10 13 4·3 E	123·0	20·0	8·2

Deimos.

1888. G.M.T. d h m	p °	a ″	b ″	1888. G.M.T. d h m	p °	a ″	b ″
Mar. 20 15 43·4 E	125·1	47·8	15·5	Apr. 23 16 47·0 E			
22 13 7·9 W				25 14 10·7 W			
24 10 32·3 E				27 11 34·4 E	122·1	52·6	20·1
26 7 56·5 W				29 8 58·3 W			
28 5 20·5 E				May 1 6 22·4 E			
30 2 44·4 W	304·6	50·4	16·8	3 3 46·9 W			
Apr. 1 0 8·1 E				5 1 11·9 E			
2 21 31·6 W				6 22 37·2 W	301·4	51·2	20·1
4 18 54·9 E				8 20 2·8 E			
6 16 18·2 W				10 17 28·6 W			
8 13 41·4 E	124·0	52·6	18·4	12 14 54·6 E			
10 11 4·5 W				14 12 20·8 W			
12 8 27·6 E				16 9 47·1 E	121·0	50·6	20·0
14 5 50·7 W				18 7 13·4 W			
16 3 13·8 E				20 4 39·8 E			
18 0 37·0 W	302·9	53·5	19·6	22 2 6·4 W			
19 22 0·3 E				23 23 33·1 E			
21 19 23·6 W				25 21 0·0 W	300·2	48·8	19·7

Phobos.

	G.M.T.				p	a	b		G.M.T.				p	a	b
1890.	d	h	m		°	″	″	1890.	d	h	m		°	″	″
May	8	0	44·8	E	123·6	23·5	2·3	June	7	3	52·1	W			
	9	3	31·8	W					8	6	39·0	E			
	10	6	18·8	E					9	9	26·0	W			
	11	9	5·8	W					10	12	12·9	E	126·7	26·5	5·3
	12	11	52·8	E					11	14	59·9	W			
	13	14	39·8	W	304·1	25·0	3·0		12	17	46·8	E			
	14	17	26·8	E					13	20	33·8	W			
	15	20	13·8	W					14	23	20·8	E			
	16	23	0·8	E					16	2	7·7	W	307·1	26·0	5·6
	18	1	47·8	W					17	4	54·8	E			
	19	4	34·7	E	124·5	25·4	3·3		18	7	41·7	W			
	20	7	21·7	W					19	10	28·9	E			
	21	10	8·6	E					20	13	16·0	W			
	22	12	55·6	W					21	16	3·2	E	127·3	25·5	5·7
	23	15	42·5	E					22	18	50·2	W			
	24	18	29·5	W	305·0	26·0	3·8		23	21	37·4	E			
	25	21	16·4	E					25	0	24·6	W			
	27	0	3·3	W					26	3	11·9	E			
	28	2	50·2	E					27	5	59·2	W	307·5	24·9	5·7
	29	5	37·0	W					28	8	46·4	E			
	30	8	23·9	E	125·6	26·3	4·3		29	11	33·7	W			
	31	11	10·8	W					30	14	21·0	E			
June	1	13	57·6	E				July	1	17	8·3	W			
	2	16	44·6	W					2	19	55·6	E	127·7	24·1	5·7
	3	19	31·4	E					3	22	42·9	W			
	4	22	18·4	W	306·1	26·7	4·8		5	1	30·2	E			
	6	1	5·2	E					6	4	17·6	W			

Deimos.

	G.M.T.				p	a	b		G.M.T.				p	a	b
1890.	d	h	m		°	″	″	1890.	d	h	m		°	″	″
Apr.	30	22	40·9	E	122·5	55·1	5·6	May	16	2	3·6	E			
May	2	20	6·9	W					17	23	28·0	W			
	4	17	32·7	E					19	20	52·3	E	124·0	63·7	9·4
	6	14	58·3	W					21	18	16·6	W			
	8	12	23·6	E					23	15	40·9	E			
	10	9	48·9	W	303·1	59·7	7·2		25	13	5·1	W			
	12	7	14·0	E					27	10	29·1	E			
	14	4	38·9	W					29	7	53·1	W	304·9	66·1	11·8

1890. d	G.M.T. h	m		p	a	b	1890. d	G.M.T. h	m		p	a	b
				°	″	″					°	″	′
May 31	5	17·1	E				June 19	3	21·4	E			
June 2	2	41·1	W				21	0	46·7	W			
4	0	5·3	E				22	22	12·3	E			
5	21	29·4	W				24	19	38·1	W			
7	18	53·6	E	125·6	66·6	13·8	26	17	4·1	E	126·5	62·1	15·2
9	16	17·9	W				28	14	30·3	W			
11	13	42·4	E				30	11	56·6	E			
13	11	6·9	W				July 2	9	23·1	W			
15	8	31·5	E				4	6	49·8	E			
17	5	56·3	W	306·1	65·6	14·8	6	4	16·8	W	306·6	57·6	14·5

In these ephemerides p denotes the position-angle of the major axis of the satellite's apparent orbit, and is reckoned from the north towards the east, and from 0° to 360°; a and b denote the major and minor semi-axes of the apparent orbit. During these two oppositions of the planet the satellites move in the direction of *increasing* position-angles, the Earth being *above* the plane of the orbits. The time of greatest elongation has been given to the nearest tenth of a minute in order that a sufficiently accurate comparison may be had with observation.

Spottiswoode & Co. Printers, New-street Square, London.

Barnard's Comet.

From the Observations Windsor on July 23, at McCabe Sept 17 and at Nice + Vienna on Oct 24, the following Elements result,

T	1884 Aug 16.268808 W. M. T.			
ω	301°	2′	42″.4	Mean Eq. 1884.0
Ω	5	8	37.5	
i	5	27	33.2	
φ	35	43	44.8	
log q	0.1071047			
log a	0.4879630			
μ	657″.752			
P	1970.345 Days.			